Curvature Diffusions in General Relativity

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Abstract

We define and study on Lorentz manifolds a family of covariant diffusions in which the quadratic variation is locally determined by the curvature. This allows the interpretation of the diffusion effect on a particle by its interaction with the ambient space-time. We will focus on the case of warped products, especially Robertson-Walker manifolds, and analyse their asymptotic behaviour in the case of Einstein-de Sitter-like manifolds.

Contents

1 Introduction

It is known since Dudley's pioneer's work [Du] that a relativistic diffusion, i.e. a Lorentz-covariant Markov diffusion process, cannot exist on the base space, even in the Minkowski framework of special relativity, but possibly makes sense at the level of the tangent bundle. In this spirit, the general case of a Lorentz manifold was first investigated in [F-LJ], where a general relativistic diffusion was introduced. The quadratic variation of this diffusion is constant and does not vanish in the vacuum. In this article, we investigate diffusions whose quadratic variation is locally determined by the curvature of the space, and then vanishes in empty (or at least flat) regions.

The relativistic diffusion considered in [F-LJ] lives on the pseudo-unit tangent bundle $T^1\mathcal{M}$ of the given generic Lorentz manifold $(\mathcal{M}, g)$. As is recalled in Section 3.1 below, it can be obtained by superposing, to the geodesic flow of $T^1\mathcal{M}$, random fluctuations of the velocity that are given by hyperbolic Brownian motion, if one identifies the tangent space $T^1_\xi\mathcal{M}$ with the hyperbolic space $\mathbb{H}^d$ (at point $\xi \in \mathcal{M}$, by means of the pseudo-metric g). These Brownian fluctuations of the velocity can be defined by the vertical Dirichlet form $\int_{T^1\mathcal{M}} \left|\nabla^v_{\dot\xi} F(\xi,\dot\xi)\right|^2 \mu(d\xi, d\dot\xi)$, considered with respect to the Liouville measure μ.

The Dirichlet forms we investigate in this article depend only on the local geometry of $(\mathcal{M}, g)$, e.g. on the curvature tensor at the current point ξ, and on the velocity $\dot\xi$. We consider several examples:

- If the scalar curvature $R(\xi)$ is everywhere non-positive (which is physically relevant, see [L-L]), then the Dirichlet form can be: $-\int_{T^1\mathcal{M}} \left|\nabla^v_{\dot\xi} F(\xi,\dot\xi)\right|^2 R(\xi)\, \mu(d\xi, d\dot\xi)$,

leading to the covariant relativistic diffusion we call <u>R-diffusion</u>.

- If the energy $\mathcal{E}(\xi,\dot\xi)$ is everywhere non-negative (which is physically relevant, see [L-L], and [H-E], where this is called the "weak energy condition"), then we can choose the Dirichlet form to be: $\int_{T^1\mathcal{M}} \left|\nabla^v_{\dot\xi} F(\xi,\dot\xi)\right|^2 \mathcal{E}(\xi,\dot\xi)\, \mu(d\xi, d\dot\xi)$,

leading to another covariant relativistic diffusion, we call <u>energy $\mathcal{E}$-diffusion</u>.

Contrary to the basic relativistic diffusion, these new relativistic diffusions reduce to the geodesic flow in every empty (vacuum) region.

Note that $-R(\xi)$ and $\mathcal{E}(\xi,\dot\xi)$ could be replaced by $\varphi(R(\xi))$ and $\psi(\mathcal{E}(\xi,\dot\xi)$, for more or less arbitrary non-negative increasing functions φ, ψ. We shall present this class of <u>covariant Ξ-relativistic diffusions</u>, or <u>Ξ-diffusions</u>, in Section 3 below.

- If the sectional curvatures of timelike planes are everywhere non-negative (sectional curvature has proved to be a natural tool in Lorentzian geometry, see for example [H], [H-R]), as this is often the case (at least in usual symmetrical examples), then it is possible to construct a covariant sectional relativistic diffusion which undergoes velocity fluctuations that are no longer isotropically Brownian, using the whole curvature tensor (not the Ricci tensor alone). See Section 6 below. This sectional relativistic diffusion depends on the curvature tensor in a canonical way. Its diffusion symbol vanishes in flat regions (i.e. regions where the whole curvature tensor vanishes), but does not vanish, in general, in empty regions (i.e. regions where the Ricci tensor vanishes).

Note that all these covariant diffusions are the projections on $T^1\mathcal{M}$ of diffusions on the frame bundle $G(\mathcal{M})$. Actually, they are constructed directly on $G(\mathcal{M})$, as in [F-LJ] and in the classical construction of Brownian motion on Riemannian manifolds, see [El], [M], [I-W], [Em], [Hs], [A-C-T]. These constructions are performed in Sections 3 and 6 below.

Note also that, while in the flat case the Dudley diffusion [Du] is the unique covariant diffusion, the above examples show that this is not at all the same for curved spaces.

In Sections 4 and 5, we study in more detail the case of warped products, and specify further some particularly symmetrical examples, namely Robertson-Walker manifolds, which are warped products with energy-momentum tensor of perfect fluid type.

We investigate more closely Einstein-de Sitter-like manifolds (Robertson-Walker manifolds for which the expansion rate is $\alpha(t) = t^c$ for some positive c), reviewing in this simple class of examples, the relativistic diffusions we introduced, which appear to be distinct. We perform in this setting an asymptotic study of the R-diffusion, and of the minimal sub-diffusion $(t_s, \dot{t}_s)$ relative to the energy $\mathcal{E}$-diffusion.

2 Canonical vector fields and curvature

We present in this section the main notations and recall a few known facts (see [K-N]).

2.1 Isomorphism between $\bigwedge^2 \mathbb{R}^{1,d}$ and $so(1,d)$

On the Minkowski space-time $\mathbb{R}^{1,d}$, we denote by $\eta = ((\eta_{ij}))_{0\le i,j\le d}$ the Minkowski tensor $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & -1 \end{pmatrix}$. We also denote by $((\eta^{ij}))_{0\le i,j\le d}$ the inverse tensor, so that $\eta_{ij}\eta^{jk} = \delta_i^k$ (or equivalently: $\eta_{ij} = \eta^{ij} := 1_{\{i=j=0\}} - 1_{\{1\le i=j\le d\}}$), and by $\langle\cdot,\cdot\rangle_\eta$ the corresponding Minkowski pseudo-metric. For $u, v, w \in \mathbb{R}^{1,d}$, we set

$$u \wedge v\,(w) := \langle u, w\rangle_\eta\, v - \langle v, w\rangle_\eta\, u\,. \tag{1}$$

In other terms, this is the interior product of $u \wedge v$ by the dual of w with respect to η. This defines an endomorphism of $\mathbb{R}^{1,d}$ which belongs to $so(1,d)$, since for any $w, w' \in \mathbb{R}^{1,d}$ we have clearly $\langle u \wedge v(w), w' \rangle_\eta + \langle w, u \wedge v(w') \rangle_\eta = 0$. It vanishes only if u and v are collinear, hence if and only if $u \wedge v = 0$. We have thus an isomorphism between $\bigwedge^2 \mathbb{R}^{1,d}$ and $so(1,d)$.

Remark 2.1.1 *The Lie bracket of $so(1,d)$ can be expressed, for any $a, b, u, v \in \mathbb{R}^{1,d}$, by:*

$$[a \wedge b, u \wedge v] = \langle a, u \rangle_\eta \, b \wedge v + \langle b, v \rangle_\eta \, a \wedge u - \langle a, v \rangle_\eta \, b \wedge u - \langle b, u \rangle_\eta \, a \wedge v .$$

The Minkowski pseudo-metric $\langle \cdot, \cdot \rangle_\eta$ extends to $\bigwedge^2 \mathbb{R}^{1,d}$, by setting:

$$\langle u \wedge v, a \wedge b \rangle_\eta := \langle u, a \rangle_\eta \, \langle v, b \rangle_\eta - \langle u, b \rangle_\eta \, \langle v, a \rangle_\eta = \tfrac{1}{2} \big(\langle u \wedge v(a), b \rangle_\eta - \langle u \wedge v(b), a \rangle_\eta \big), \tag{2}$$

so that, if $(e_0, \ldots, e_d)$ is a Lorentz (i.e. pseudo-orthonormal) basis of $(\mathbb{R}^{1,d}, \eta)$, then $(e_i \wedge e_j \,|\, 0 \le i < j \le d)$ is an orthogonal basis of $(\bigwedge^2 \mathbb{R}^{1,d}, \eta)$, such that $\langle e_i \wedge e_j \,, e_i \wedge e_j \rangle_\eta = \eta_{ii} \, \eta_{jj}$.

2.2 Frame bundle $G(\mathcal{M})$ over $(\mathcal{M}, g)$

Let $\mathcal{M}$ be a time-oriented C^∞ $(1+d)$-dimensional Lorentz manifold, with pseudo-metric g having signature $(+, -, \ldots, -)$, and let $T^1\mathcal{M}$ denote the positive half of the pseudo-unit tangent bundle. Let $G(\mathcal{M})$ be the bundle of direct pseudo-orthonormal frames, with first element in $T^1\mathcal{M}$, which has its fibers modelled on the special Lorentz group. Let $\pi_1 : u \mapsto \big(\pi(u), e_0(u)\big)$ denote the canonical projection from $G(\mathcal{M})$ onto the unit tangent bundle $T^1\mathcal{M}$, which to each frame $\big(e_0(u), \ldots, e_d(u)\big)$ associates its first vector e_0.

We denote by $T\mathcal{M} \xrightarrow{\pi_2} \mathcal{M}$ the tangent bundle, by $\Gamma(TM)$ the set of C^2 vector fields on $\mathcal{M}$ (sections of π_2), by $G(\mathcal{M}) \xrightarrow{\pi} \mathcal{M}$ the frame bundle, by $u = \big(\pi(u); e_0(u), \ldots, e_d(u)\big)$ the generic element of $G(\mathcal{M})$. We extend (1) to a linear action of $so(1,d) \equiv \bigwedge^2 \mathbb{R}^{1,d}$ on $G(\mathcal{M})$, by setting:

$$e_k \wedge e_\ell \,(e_j(u)) \;:=\; \eta_{jk} \, e_\ell(u) - \eta_{j\ell} \, e_k(u) \,, \qquad \text{for any } \; 0 \le j, k, \ell \le d \,,$$

where $(e_0, \ldots, e_d)$ denotes the canonical basis of $\mathbb{R}^{1,d}$.

The action of $SO(d)$ on $(e_1, \ldots, e_d)$ induces the identification $T^1\mathcal{M} \equiv G(\mathcal{M})/SO(d)$.

The right action of $so(1,d)$ on $G(\mathcal{M})$ defines a linear map V from $so(1,d)$ into vector fields on $G(\mathcal{M})$ (i.e. sections of the canonical projection of $TG(\mathcal{M})$ on $G(\mathcal{M})$), such that

$$[V_{a\wedge b}, V_{\alpha\wedge\beta}] = V_{[a\wedge b, \alpha\wedge\beta]} \,, \qquad \text{for any } \; a \wedge b \,, \alpha \wedge \beta \in \bigwedge^2 \mathbb{R}^{1,d}. \tag{3}$$

Vector fields $V_{a\wedge b}$ are called <u>vertical</u>.

Notation To abreviate the notations, we shall consider mostly the canonical vector fields:

$$V_{ij} := V_{e_i \wedge e_j} \,, \qquad \text{for } \; 0 \le i, j \le d \,.$$

By (3) and (1), for $0 \le i,j,k,\ell \le d$ we have:

$$[V_{ij}, V_{k\ell}] = \eta_{ik}\, V_{j\ell} + \eta_{j\ell}\, V_{ik} - \eta_{i\ell}\, V_{jk} - \eta_{jk}\, V_{i\ell}\,. \tag{4}$$

We shall often write V_j for V_{0j}.

Denote by p the canonical projection $TT\mathcal{M} \xrightarrow{p} T\mathcal{M}$, by $\tilde{p}$ the canonical projection $TG(\mathcal{M}) \xrightarrow{\tilde{p}} G(\mathcal{M})$, and consider also the projection $TT\mathcal{M} \xrightarrow{(p,T\pi_2)} T\mathcal{M} \oplus T\mathcal{M}$, where

$$T\mathcal{M} \oplus T\mathcal{M} := \big\{(\xi\,; v_1, v_2) \,\big|\, \xi \in \mathcal{M}\,, v_i \in T_\xi\mathcal{M}\big\} \equiv \big\{(w_1, w_2) \,\big|\, w_i \in T\mathcal{M}\,,\ \pi_2(w_1) = \pi_2(w_2)\big\}$$

is the so-called Whitney sum.

A <u>connection</u> σ can be defined as a bilinear section $T\mathcal{M} \oplus T\mathcal{M} \xrightarrow{\sigma} TT\mathcal{M}$ of $(p, T\pi_2)$, the bilinearity being that of $(v_1, v_2) \mapsto \sigma(\xi\,; v_1, v_2)$, above any given base point $\xi \in \mathcal{M}$. Given such a connection σ and a C^1 curve γ, the <u>parallel transport</u> $/\!/_t^\gamma$ along γ of any $v_0 \in T_{\gamma_0}\mathcal{M}$ is $v_t = /\!/_t^\gamma v_0 \in T_{\gamma_t}\mathcal{M}$ defined by the ordinary differential equation: $\frac{d}{dt}(\gamma_t\,; v_t) = \sigma(\gamma_t\,; v_t\,, \dot{\gamma}_t)$. Then the <u>covariant derivative</u> $\nabla_{\dot{\gamma}_0} X(\gamma_0) \in T_{\gamma_0}\mathcal{M}$ of a C^1 vector field X is defined as the derivative at 0 of $t \mapsto \big(/\!/_t^\gamma\big)^{-1} X(\gamma_t)$.

A connection σ is said to be <u>metric</u> if the associated parallel transport preserves the pseudo-metric, and then acts on $G(\mathcal{M})$ as well as on $T\mathcal{M}$.

A metric connection σ defines the <u>horizontal vector fields</u> H_k on $G(\mathcal{M})$, for $0 \le k \le d$, given for any $F \in C^1(G(\mathcal{M}))$ and $u \in G(\mathcal{M})$, by:

$$H_k F(u) \ \text{ is the derivative at } 0 \text{ of } \ t \mapsto F\big(/\!/_t^\gamma u\big), \tag{5}$$

the C^1 curve γ being such that $\gamma_0 = \pi(u), \dot{\gamma}_0 = e_k(u)$. Note that $T\pi(H_k) = e_k$.

The canonical vectors V_{ij}, H_k span $TG(\mathcal{M})$ (the horizontal (resp. vertical) sub-bundle of $TG(\mathcal{M})$ being spanned by H_k's (resp. V_{ij}'s)). Note that H_0 generates the geodesic flow, that $V_1, \ldots, V_d$ generate the <u>boosts</u>, and that the V_{ij} $(1 \le i,j \le d)$ generate rotations.

This allows to define the intrinsic <u>torsion</u> tensor $((\mathcal{T}_{ij}^k))$ and <u>curvature</u> tensor $((\mathcal{R}_{ij}{}^{k\ell}))$ (with $0 \le i,j,k,\ell \le d$) of the metric connection σ, by the assignment:

$$[H_i, H_j] = \sum_{k=0}^{d} \mathcal{T}_{ij}^k\, H_k + \sum_{0 \le k < \ell \le d} \mathcal{R}_{ij}{}^{k\ell}\, V_{k\ell} \tag{6}$$

we can denote more simply by $\mathcal{T}_{ij}^k\, H_k + \frac{1}{2}\, \mathcal{R}_{ij}{}^{k\ell}\, V_{k\ell}$.

For any metric connection we have:

$$[V_{ij}, H_k] = \eta_{ik}\, H_j - \eta_{jk}\, H_i\,, \qquad \text{for } 0 \le i,j,k \le d\,. \tag{7}$$

There exists a unique metric connection with vanishing torsion, called the

Levi-Civita connection. We shall henceforth consider this one.

The curvature operator $\mathcal{R}_\xi$ is defined on $\bigwedge^2 T_\xi\mathcal{M}$ by:

$$\mathcal{R}_\xi\big(e_i(u)\wedge e_j(u)\big) := \sum_{0\le k<\ell\le d} \mathcal{R}_{ij}{}^{k\ell}\, e_k(u)\wedge e_\ell(u), \text{ for any } u\in\pi^{-1}(\xi) \text{ and } 0\le i,j\le d\,. \quad (8)$$

The curvature operator is alternatively given by: for any C^1 vector fields X,Y,Z,A,

$$\langle \mathcal{R}\,(X\wedge Y)\,, A\wedge Z\rangle_\eta = \Big\langle \big([\nabla_X\,,\nabla_Y] - \nabla_{[X,Y]}\big)\,Z\,,\,A\Big\rangle_g\,. \quad (9)$$

The Ricci tensor and Ricci operator are defined, for $0\le i,k\le d$, by:

$$R_i^k := \sum_{j=0}^d \mathcal{R}_{ij}{}^{kj}\,, \quad \text{and} \quad \mathrm{Ricci}_\xi\big(e_i(u)\big) := \sum_{k=0}^d R_i^k\, e_k(u)\,, \text{ for any } u\in\pi^{-1}(\xi)\,. \quad (10)$$

The scalar curvature is: $\quad R := \sum_{k=0}^d R_k^k\,.$

The indexes of the curvature tensor $(\!(\mathcal{R}_{ij}{}^{k\ell})\!)$ and of the Ricci tensor $(\!(R_i^k)\!)$ are lowered or raised by means of the Minkowski tensor $(\!(\eta_{ab})\!)$ and its inverse $(\!(\eta^{ab})\!)$. For example, we have: $\mathcal{R}^p{}_{jqr} = \mathcal{R}_{ij}{}^{k\ell}\,\eta^{ip}\,\eta_{kq}\,\eta_{\ell r}$, and $R_{ij} = R_i^k\,\eta_{kj}$.

Remark 2.2.1 The curvature and Ricci operators and tensors are symmetrical:

$$\langle \mathcal{R}(a\wedge b), v\wedge w\rangle_\eta = \langle a\wedge b, \mathcal{R}(v\wedge w)\rangle_\eta\,, \text{ and } \langle \mathrm{Ricci}(v), w\rangle_\eta = \langle v, \mathrm{Ricci}(w)\rangle_\eta\,,$$

for any $a,b,v,w\in\mathbb{R}^{1,d}$. Equivalently, for $0\le i,j,k,\ell\le d$: $R_{ij}{}^{k\ell} = R^{k\ell}{}_{ij}$, and $R_{ij} = R_{ji}$.

The energy-momentum tensor $(\!(T_j^k)\!)$ and operator T_ξ are defined as:

$$T_j^k := R_j^k - \tfrac{1}{2}\,R\,\delta_j^k \quad \text{and} \quad T_\xi := \mathrm{Ricci}_\xi - \tfrac{1}{2}\,R\,. \quad (11)$$

Note that $\sum_{j=0}^d T_j^j = -\frac{d-1}{2}\,R\,.$ The energy at any line-element $(\xi,\dot\xi)\in T^1\mathcal{M}$ is

$$\mathcal{E}(\xi,\dot\xi) := \langle T_\xi(\dot\xi),\dot\xi\rangle_{g(\xi)} = T_{00}(\xi,\dot\xi)\,. \quad (12)$$

The last equality is easily derived from (10) and (11) since writing $(\xi,\dot\xi) = (\pi(u), e_0(u))$ for any $u\in\pi_1^{-1}(\xi,\dot\xi)$ and $T_{ij} = T_i^k\,\eta_{kj}$, we have:

$$\langle T_\xi(\dot\xi),\dot\xi\rangle_{g(\xi)} = g\big(T_\xi(e_0), e_0\big) = g(T_0^k\, e_k, e_0) = T_0^k\,\eta_{k0} = T_{00} = T_{00}(\xi,\dot\xi)\,.$$

The weak energy condition (see [H-E]) stipulates that $\mathcal{E}(\xi,\dot\xi)\ge 0$ on the whole $T^1\mathcal{M}$.

We shall need the following general computation rule.

Lemma 2.2.2 *For* $0 \le i,j,k,\ell,p,q \le d$, *we have:*

$$V_{qp}\mathcal{R}_{ij}{}^{k\ell} = \eta_{qi}\,\mathcal{R}_{pj}{}^{k\ell} - \eta_{ip}\,\mathcal{R}_{qj}{}^{k\ell} + \eta_{qj}\,\mathcal{R}_{ip}{}^{k\ell} - \eta_{jp}\,\mathcal{R}_{iq}{}^{k\ell} + \delta_q^k\,\mathcal{R}_{ijp}{}^{\ell} - \delta_p^k\,\mathcal{R}_{ijq}{}^{\ell} - \delta_q^\ell\,\mathcal{R}_{ijp}{}^{k} + \delta_p^\ell\mathcal{R}_{ijq}{}^{k}.$$

Proof Using (8) and (1), we have indeed:

$$= \eta_{qi}\,\mathcal{R}_{pj}{}^{k\ell} - \eta_{ip}\,\mathcal{R}_{qj}{}^{k\ell} + \eta_{qj}\,\mathcal{R}_{ip}{}^{k\ell} - \eta_{jp}\,\mathcal{R}_{iq}{}^{k\ell} + \delta_q^k\,\mathcal{R}_{ijp}{}^{\ell} - \delta_p^k\,\mathcal{R}_{ijq}{}^{\ell} - \delta_q^\ell\,\mathcal{R}_{ijp}{}^{k} + \delta_p^\ell\mathcal{R}_{ijq}{}^{k}. \ \diamond$$

2.3 Expressions in local coordinates

Consider local coordinates (ξ^i, e_j^k) for $u = (\xi, e_0, \dots, e_d) \in G(\mathcal{M})$, with $e_j = e_j^k \dfrac{\partial}{\partial \xi^k}\cdot$

Then the horizontal and vertical vector fields V_{ij}, V_j, H_k , which satify the commutation relations (4),(6),(7) of the preceding section 2.2, read as follows. First, denoting by $\Gamma_{kj}^\ell = \Gamma_{jk}^\ell$ the Christoffel coefficients of the Levi-Civita connexion ∇, we have for $0 \le i,j \le d$:

$$\nabla_{\frac{\partial}{\partial \xi^i}} \frac{\partial}{\partial \xi^j} = \Gamma_{ij}^k(\xi)\,\frac{\partial}{\partial \xi^k} \qquad \text{and} \qquad H_j = e_j^k\,\frac{\partial}{\partial \xi^k} - e_j^k\,e_i^m\,\Gamma_{km}^\ell(\xi)\,\frac{\partial}{\partial e_i^\ell}\cdot \tag{13}$$

This is consistent with (5). Indeed, we have a priori an expression $H_j = a_j^k \frac{\partial}{\partial \xi^k} + b_{ji}^\ell \frac{\partial}{\partial e_i^\ell}$, with on one hand $a_j^k = \langle T\pi(H_j), d\xi^k\rangle = \langle e_j, d\xi^k\rangle = e_j^k$. On the other hand, for a C^1 curve γ satisfying $\gamma_0 = \xi\,, \dot\gamma_0 = e_j$, denoting by e^{-1} the matrix inverse to $e \equiv (\!(e_i^k)\!)$ we have:

$$e_j^k\,\Gamma_{km}^\ell\,\frac{\partial}{\partial \xi^\ell} = \nabla_{\dot\gamma_0}\frac{\partial}{\partial \xi^m} = \nabla_{\dot\gamma_0}\big((e^{-1})_m^i\,e_i\big) = \frac{d_o}{dt}\Big[\big(/\!/_t^{\gamma}\big)^{-1}\big((e^{-1})_m^i\,e_i\big)\big(/\!/_t^{\gamma}u\big)\Big]$$

$$= \frac{d_o}{dt}\Big[(e^{-1})_m^i\big(/\!/_t^{\gamma}u\big)\Big]e_i = H_j\big((e^{-1})_m^i\big)\,e_i^\ell\,\frac{\partial}{\partial \xi^\ell} = -(e^{-1})_m^i\,H_j(e_i^\ell)\,\frac{\partial}{\partial \xi^\ell}$$

by (5), so that $b_{ji}^\ell = H_j(e_i^\ell) = -e_j^k\,e_i^m\,\Gamma_{km}^\ell$ as wanted.

Recall that the Christoffel coefficients of ∇ are computed by:

$$\Gamma_{ij}^k = \tfrac{1}{2}\,g^{k\ell}\Big(\frac{\partial g_{\ell j}}{\partial \xi^i} + \frac{\partial g_{i\ell}}{\partial \xi^j} - \frac{\partial g_{ij}}{\partial \xi^\ell}\Big)\,,$$

or equivalently, by the fact that geodesics solve $\ddot\xi^k + \Gamma_{ij}^k\dot\xi^i\dot\xi^j = 0$.

Then for $0 \le i,j,k \le d$:

$$V_{ij}\,e_k = \eta_{ik}\,e_j - \eta_{jk}\,e_i = (\eta_{ik}\,e_j^\ell - \eta_{jk}\,e_i^\ell)\,\frac{\partial}{\partial \xi^\ell} = (\eta_{iq}\,e_j^m - \eta_{jq}\,e_i^m)\,\frac{\partial}{\partial e_q^m}\,e_k^\ell\,\frac{\partial}{\partial \xi^\ell}\,,$$

whence for $0 \le i,j \le d$:

$$V_{ij} \;=\; (\eta_{iq}\,e_j^m - \eta_{jq}\,e_i^m)\,\frac{\partial}{\partial e_q^m}\,, \tag{14}$$

that is: $\quad V_{ij} = e_i^k \dfrac{\partial}{\partial e_j^k} - e_j^k \dfrac{\partial}{\partial e_i^k}$ and $V_j = e_0^k \dfrac{\partial}{\partial e_j^k} + e_j^k \dfrac{\partial}{\partial e_0^k}$, for $1 \le i, j \le d$.

The curvature operator expresses in a local chart as: for $0 \le m, n, p, q \le d$,

$$\widetilde{\mathcal{R}}_{mnpq} := \left\langle \mathcal{R}\left(\tfrac{\partial}{\partial \xi^m} \wedge \tfrac{\partial}{\partial \xi^n}\right), \tfrac{\partial}{\partial \xi^p} \wedge \tfrac{\partial}{\partial \xi^q} \right\rangle_g = g_{mr}\left(\Gamma^r_{ps}\,\Gamma^s_{nq} - \Gamma^r_{qs}\,\Gamma^s_{np} + \frac{\partial \Gamma^r_{nq}}{\partial \xi^p} - \frac{\partial \Gamma^r_{np}}{\partial \xi^q}\right). \quad (15)$$

Then, the Ricci operator can be computed similarly, as: for $0 \le m, p \le d$,

$$\tilde{R}_{mp} := \left\langle \mathrm{Ricci}(\tfrac{\partial}{\partial \xi^m}), \tfrac{\partial}{\partial \xi^p} \right\rangle_g = \widetilde{\mathcal{R}}_{mnpq}\, g^{nq} = \Gamma^n_{nq}\Gamma^q_{mp} - \Gamma^n_{pq}\,\Gamma^q_{mn} + \frac{\partial \Gamma^n_{mp}}{\partial \xi^n} - \frac{\partial \Gamma^n_{mn}}{\partial \xi^p}\,. \quad (16)$$

The scalar curvature and the energy-momentum operator can be computed by:

$$R = \tilde{R}_{ij}\, g^{ij} \quad \text{and} \quad \tilde{T}_{\ell m} = \tilde{R}_{\ell m} - \tfrac{1}{2}\, R\, g_{\ell m}\,. \quad (17)$$

To summarize, the Riemann curvature tensor $(\!(\mathcal{R}_{ij}{}^{k\ell})\!)$ is made of the coordinates of the curvature operator $\mathcal{R}$ in an orthonormal moving frame, and its indexes are lowered or raised by means of the Minkowski tensor $(\!(\eta_{ab})\!)$, while the curvature tensor $(\!(\tilde{R}_{mnpq})\!)$ is made of the coordinates of the curvature operator in a local chart, and its indexes are lowered or raised by means of the metric tensor $(\!(g_{ab})\!)$.

To go from one tensor to the other, note that by (15) and (8) we have $\mathcal{R}\big(\frac{\partial}{\partial \xi^m} \wedge \frac{\partial}{\partial \xi^n}\big) = \frac{1}{2}\, \widetilde{\mathcal{R}}_{mn}{}^{ab}\, \frac{\partial}{\partial \xi^a} \wedge \frac{\partial}{\partial \xi^b}$, whence: $e_i^k\, e_j^\ell\, \widetilde{\mathcal{R}}_{k\ell}{}^{pq} = \mathcal{R}_{ij}{}^{mn}\, e_m^p\, e_n^q$, or equivalently:

$$\mathcal{R}_{ijab} = \widetilde{\mathcal{R}}_{k\ell rs}\, e_i^k\, e_j^\ell\, e_a^r\, e_b^s\,, \qquad \text{or as well:} \quad \widetilde{\mathcal{R}}^{rspq} = \mathcal{R}^{abmn}\, e_a^r\, e_b^s\, e_m^p\, e_n^q\,. \quad (18)$$

2.4 Case of a perfect fluid

The energy-momentum tensor T (of (11), or equivalently $\tilde{T}$, recall (17)) is associated to a perfect fluid (see [H-E]) if it has the form :

$$\tilde{T}_{k\ell} = q\, U_k\, U_\ell - p\, g_{k\ell}\,, \quad (19)$$

for some C^1 field U in $T^1\mathcal{M}$ (which represents the velocity of the fluid), and some C^1 functions p, q on $\mathcal{M}$. By Einstein Equations (17), (19) is equivalent to:

$$\tilde{R}_{k\ell} = q\, U_k\, U_\ell + \tilde{p}\, g_{k\ell}\,, \quad \text{with} \quad \tilde{p} = (2p - q)/(d-1), \quad (20)$$

or as well, by (16), to:

$$\langle \mathrm{Ricci}(W), W \rangle_\eta = q \times g(U, W)^2 + \tilde{p} \times g(W, W)\,, \quad \text{for any } W \in T\mathcal{M}\,. \quad (21)$$

The quantity $\langle U(\xi_s), \dot{\xi}_s \rangle$, (which is the hyperbolic cosine of the distance, on the unit hyperboloid at ξ_s identified with the hyperbolic space between the space-time velocities of the fluid and the path) will be denoted by $\mathcal{A}_s$ or $\mathcal{A}(\xi_s, \dot{\xi}_s)$.

Note that necessarily $\mathcal{A}_s \ge 1$. By Formulas (12) and (19), the energy equals:

$$\mathcal{E}(\xi, \dot{\xi}) = q(\xi)\, \mathcal{A}(\xi, \dot{\xi})^2 - p(\xi). \quad (22)$$

Remark 2.4.1 (i) The energy of the fluid is simply: $\tilde{T}_{k\ell}\, U^k\, U^\ell = q - p\,.$ and the scalar curvature equals $R = 2\,[(d+1)\,p - q]/(d-1)$.

(ii) By (22), the weak energy condition reads here: $q \geq p^+$.

3 Covariant Ξ-relativistic diffusions

Let Ξ denote a non-negative smooth function on $G(\mathcal{M})$, invariant under the right action of $SO(d)$ (so that it identifies with a function on $T^1\mathcal{M}$).

Our examples will be $\Xi = -\,\varrho^2\, R$ and $\Xi = \varrho^2\, \mathcal{E}$ (for a positive constant ϱ).

We call Ξ-relativistic diffusion or Ξ-diffusion the $G(\mathcal{M})$-valued diffusion process (Φ_s) associated to Ξ we will construct in Section 3.2 below, as well as its $T^1\mathcal{M}$-valued projection $\pi_1(\Phi_s)$. Let us first recall our previous construction, which corresponds to a constant Ξ.

3.1 The basic relativistic diffusion

The relativistic diffusion process $(\xi_s, \dot{\xi}_s)$ was defined in [F-LJ], as the projection under π_1 of the $G(\mathcal{M})$-valued diffusion (Ψ_s) solving the following Stratonovitch stochastic differential equation (for a given $\mathbb{R}^d$-valued Brownian motion (w_s^j) and some fixed $\varrho > 0$):

$$d\Psi_s = H_0(\Psi_s)\, ds + \varrho \sum_{j=1}^{d} V_j(\Psi_s) \circ dw_s^j\,. \tag{23}$$

The infinitesimal generator of the $G(\mathcal{M})$-valued relativistic diffusion (Ψ_s) is

$$\mathcal{H} \,:=\, H_0 + \tfrac{\varrho^2}{2} \sum_{j=1}^{d} V_j^2\,, \tag{24}$$

and the infinitesimal generator of the relativistic diffusion $(\xi_s, \dot{\xi}_s) := \pi_1(\Psi_s)$ is the relativistic operator:

$$\mathcal{H}^1 := \mathcal{L}^0 + \tfrac{\varrho^2}{2}\,\Delta^v \,=\, \dot{\xi}^k \frac{\partial}{\partial \xi^k} + \Big(\tfrac{d\,\varrho^2}{2}\,\dot{\xi}^k - \dot{\xi}^i \dot{\xi}^j\, \Gamma_{ij}^k(\xi)\Big) \frac{\partial}{\partial \dot{\xi}^k} + \tfrac{\varrho^2}{2}\,(\dot{\xi}^k \dot{\xi}^\ell - g^{k\ell}(\xi))\, \frac{\partial^2}{\partial \dot{\xi}^k \partial \dot{\xi}^\ell}\,, \tag{25}$$

where $\mathcal{L}^0$ denotes the vector field on $T^1\mathcal{M}$ generating the geodesic flow, and Δ^v denotes the vertical Laplacian, i.e. the Laplacian on $T^1_{\dot{\xi}}\mathcal{M}$ equipped with the hyperbolic metric induced by $g(\xi)$. The relativistic diffusion $(\xi_s, \dot{\xi}_s)$ is parametrized by proper time $s \geq 0$, possibly till some positive explosion time.

In local coordinates (ξ^i, e_j^k), setting $\Psi_s = (\xi_s^i, e_j^k(s))$, Equation (23) becomes locally equivalent to the following system of Itô equations:

$$d\xi_s^k = \dot{\xi}_s^k\, ds = e_0^k(s)\, ds\;;\;\; d\dot{\xi}_s^k = -\Gamma_{i\ell}^k(\xi_s)\, \dot{\xi}_s^i\, \dot{\xi}_s^\ell\, ds + \varrho \sum_{i=1}^{d} e_i^k(s)\, dw_s^i + \tfrac{d\,\varrho^2}{2}\, \dot{\xi}_s^k\, ds\,, \quad \text{and}$$

$$de_j^k(s) = -\Gamma_{i\ell}^k(\xi_s)\, e_j^\ell(s)\, \dot\xi_s^i\, ds + \varrho\, \dot\xi_s^k\, dw_s^j + \tfrac{\varrho^2}{2}\, e_j^k(s)\, ds\,, \qquad \text{for } 1 \le j \le d\,,\ 0 \le k \le d\,.$$

Remark 3.1.1 We have on $T^1\mathcal{M}$:

$$\sum_{j=1}^d V_j^2 \mathcal{E} = 2(d+1)\,\mathcal{E} - 2\,\mathrm{Tr}(T) = 2(d+1)\,\mathcal{E} + (d-1)\,R\,. \tag{26}$$

Indeed, since for each $1 \le j \le d$ V_j exchanges the basis vectors $e_0 = \dot\xi$ and e_j (recall (14)) we get: $V_j^2\mathcal{E} = 2\,V_j(\tilde T_{\ell m}\,\dot\xi^\ell\, e_j^m) = 2\,\tilde T_{\ell m}\,(\dot\xi^\ell\,\dot\xi^m + e_j^\ell\, e_j^m)$, whence

$$\sum_{j=1}^d V_j^2 \mathcal{E} = 2d\,\mathcal{E} + 2\,\tilde T_{\ell m}\,(\dot\xi^\ell\,\dot\xi^m - g^{\ell m}) = 2(d+1)\,\mathcal{E} - 2\,\mathrm{Tr}(\tilde T)\,,$$

and Formulas (26) follow at once by (11).

As an application, a direct computation yields the following evolution of the energy.

Remark 3.1.2 *The random energy process $\mathcal{E}_s = \mathcal{E}(\xi_s, \dot\xi_s)$ associated to the basic relativistic diffusion $\pi_1(\Psi_s) = (\xi_s, \dot\xi_s)$ satisfies the following equation (where $\nabla_v := v^j\nabla_j$):*

$$d\mathcal{E}_s = \nabla_{\dot\xi_s}\mathcal{E}_s\, ds + \varrho^2\left[(d+1)\mathcal{E}_s + \tfrac{d-1}{2}\, R(\xi_s)\right] ds + dM_s^{\mathcal{E}}\,,$$

with the quadratic variation of its martingale part $dM_s^{\mathcal{E}}$ given by:

$$[d\mathcal{E}_s, d\mathcal{E}_s] = [dM_s^{\mathcal{E}}, dM_s^{\mathcal{E}}] = 4\varrho^2\,[\mathcal{E}_s^2 - \langle \tilde T\dot\xi_s, \tilde T\dot\xi_s\rangle]\, ds\,.$$

(We have here in particular $\quad \nabla_{\dot\xi_s}\mathcal{E}_s = \left[\partial_{\xi^k}\tilde T_{ij}(\xi_s) - 2\,\tilde T_{i\ell}(\xi_s)\,\Gamma_{jk}^\ell(\xi_s)\right] \dot\xi_s^i\,\dot\xi_s^j\,\dot\xi_s^k$.)

Note that the energy $\mathcal{E}_s$ is not, in general, a Markov process.

3.2 Construction of the Ξ-diffusion

Let us start with the following Stratonovitch stochastic differential equation on $G(\mathcal{M})$ (for a given $\mathbb{R}^d$-valued Brownian motion (w_s^j)):

$$d\Phi_s = H_0(\Phi_s)\, ds + \tfrac14 \sum_{j=1}^d V_j\,\Xi\,(\Phi_s) V_j(\Phi_s)\, ds + \sum_{j=1}^d \sqrt{\Xi(\Phi_s)}\, V_j(\Phi_s) \circ dw_s^j\,. \tag{27}$$

Note that all coefficients in this equation are clearly smooth, except $\sqrt{\Xi}$ on its vanishing set $\Xi^{-1}(0)$. However, $\sqrt{\Xi}$ is a locally Lipschitz function; see ([I-W], Proposition IV.6.2). Hence, Equation (27) does define a unique $G(\mathcal{M})$-valued diffusion (Φ_s).

We have the following proposition, defining the Ξ-relativistic diffusion (or Ξ-diffusion) (Φ_s) on $G(\mathcal{M})$, and $(\xi_s, \dot\xi_s)$ on $T^1\mathcal{M}$, possibly till some positive explosion time.

Proposition 3.2.1 *The Stratonovitch stochastic differential equation (27) has a unique solution* $(\Phi_s) = (\xi_s\,;\,\dot\xi_s\,, e_1(s), \ldots, e_d(s))$*, possibly defined till some positive explosion time. This is a* $G(\mathcal{M})$*-valued covariant diffusion process, with generator*

$$\mathcal{H}_\Xi \;:=\; H_0 + \tfrac{1}{2}\sum_{j=1}^{d} V_j\,\Xi\,V_j\,. \tag{28}$$

Its projection $\pi_1(\Phi_s) = (\xi_s, \dot\xi_s)$ *defines a covariant diffusion on* $T^1\mathcal{M}$*, with* $SO(d)$*-invariant generator*

$$\mathcal{H}^1_\Xi \;:=\; \mathcal{L}^0 + \tfrac{1}{2}\,\nabla^v\,\Xi\,\nabla^v\,, \tag{29}$$

∇^v *denoting the gradient on* $T^1_\xi\mathcal{M}$ *equipped with the hyperbolic metric induced by* $g(\xi)$*.*

Moreover, the adjoint of $\mathcal{H}_\Xi$ *with respect to the Liouville measure of* $G(\mathcal{M})$ *is* $\mathcal{H}^*_\Xi \;:=\; -H_0 + \frac{1}{2}\sum_{j=1}^{d} V_j\,\Xi\,V_j\,.$ *In particular, if there is no explosion, then the Liouville measure is invariant. Furthermore, if* Ξ *does not depend on* $\dot\xi$*, i.e. is a function on* $\mathcal{M}$*, then the Liouville measure is preserved by the stochastic flow defined by Equation (27).*

We specify at once how this looks in a local chart, before giving a proof for both statements.

Corollary 3.2.2 *The* $T^1\mathcal{M}$*-valued* Ξ*-diffusion* $(\xi_s, \dot\xi_s)$ *satisfies* $d\xi_s = \dot\xi_s\,ds$*, and in any local chart, the following Itô stochastic differential equations: for* $0 \le k \le d$*, (denoting* $\Xi_s = \Xi(\xi_s, \dot\xi_s)$*)*

$$d\dot\xi^k_s = dM^k_s - \Gamma^k_{ij}(\xi_s)\,\dot\xi^i_s\,\dot\xi^j_s\,ds + \tfrac{d}{2}\,\Xi_s\,\dot\xi^k_s\,ds + \tfrac{1}{2}\,[\dot\xi^k_s\,\dot\xi^\ell_s - g^{k\ell}(\xi_s)]\,\frac{\partial\Xi}{\partial\dot\xi^\ell}(\xi_s, \dot\xi_s)\,ds\,, \tag{30}$$

with the quadratic covariation matrix of the martingale term (dM_s) *given by:*

$$[d\dot\xi^k_s, d\dot\xi^\ell_s] = [\dot\xi^k_s\,\dot\xi^\ell_s - g^{k\ell}(\xi_s)]\,\Xi_s\,ds\,, \quad \text{for } 0 \le k, \ell \le d\,.$$

<u>Proof</u> In local coordinates (ξ^i, e^k_j), $\Phi = (\xi\,;\,e_0\,, e_1, \ldots, e_d)$, using Section 2.3, Equation (27) reads: for any $0 \le k \le d$, $d\xi^k_s = \dot\xi^k_s\,ds = e^k_0(s)\,ds$,

$$d\dot\xi^k_s = -\Gamma^k_{i\ell}(\xi_s)\,\dot\xi^i_s\,\dot\xi^\ell_s\,ds + \tfrac{1}{4}\sum_{j=1}^{d} V_j\,\Xi\,(\xi_s, \dot\xi_s)\,e^k_j(s)\,ds + \sum_{j=1}^{d}\sqrt{\Xi(\xi_s, \dot\xi_s)}\,e^k_j(s)\circ dw^j_s\,;$$

and for $1 \le j \le d$, $0 \le k \le d$,

$$de^k_j(s) = -\Gamma^k_{i\ell}(\xi_s)\,e^\ell_j(s)\,\dot\xi^i_s\,ds + \tfrac{1}{4}\,V_j\,\Xi\,(\xi_s, \dot\xi_s)\,\dot\xi^k_s\,ds + \sqrt{\Xi(\xi_s, \dot\xi_s)}\,\dot\xi^k_s\circ dw^j_s\,.$$

We compute now the Itô corrections, which involve the partial derivatives of $\sqrt{\Xi(\xi, \dot\xi)}$ with respect to $\dot\xi$. We get successively, for $1 \le j \le d$, $0 \le k \le d$:

$$\sqrt{\Xi(\xi_s,\dot\xi_s)}\left[d\left(\sqrt{\Xi(\xi_s,\dot\xi_s)}\,e_j^k(s)\right),dw_s^j\right] = \Xi(\xi_s,\dot\xi_s)\left[de_j^k(s),\,dw_s^j\right]+\tfrac12\,e_j^k(s)\left[d\,\Xi(\xi_s,\dot\xi_s),dw_s^j\right]$$

$$= \Xi\,(\xi_s,\dot\xi_s)^{3/2}\,\dot\xi_s^k\,ds+\tfrac12\,e_j^k(s)\,\frac{\partial\,\Xi}{\partial\dot\xi^\ell}(\xi_s,\dot\xi_s)\sqrt{\Xi(\xi_s,\dot\xi_s)}\,e_j^\ell(s)\,ds\,,$$

hence,

$$\left[d\left(\sqrt{\Xi(\xi_s,\dot\xi_s)}\,e_j^k(s)\right),dw_s^j\right] = \Xi\,(\xi_s,\dot\xi_s)\,\dot\xi_s^k\,ds+\tfrac12\,V_j\,\Xi\,(\xi_s,\dot\xi_s)\,e_j^k(s)\,ds\,;$$

$$\sqrt{\Xi(\xi_s,\dot\xi_s)}\left[d\left(\sqrt{\Xi(\xi_s,\dot\xi_s)}\,\dot\xi_s^k\right),dw_s^j\right] = \Xi(\xi_s,\dot\xi_s)\left[d\dot\xi_s^k\,,\,dw_s^j\right]+\tfrac12\,\dot\xi_s^k\left[d\,\Xi(\xi_s,\dot\xi_s),dw_s^j\right]$$

$$= \Xi\,(\xi_s,\dot\xi_s)^{3/2}\,e_j^k(s)\,ds+\tfrac12\,\dot\xi_s^k\,\frac{\partial\,\Xi}{\partial\dot\xi^\ell}(\xi_s,\dot\xi_s)\sqrt{\Xi(\xi_s,\dot\xi_s)}\,e_j^\ell(s)\,ds\,,$$

hence,

$$\left[d\left(\sqrt{\Xi(\xi_s,\dot\xi_s)}\,\dot\xi_s^k\right),dw_s^j\right] = \Xi\,(\xi_s,\dot\xi_s)\,e_j^k(s)\,ds+\tfrac12\,V_j\,\Xi\,(\xi_s,\dot\xi_s)\,\dot\xi_s^k\,ds\,.$$

Note that the simplification by $\sqrt{\Xi(\xi_s,\dot\xi_s)}$ is allowed, since on $\Xi^{-1}(0)$ both sides of the simplified formula vanish identically.

Hence, in local coordinates and in Itô form, Equation (27) reads: for any $0\le k\le d$, $d\xi_s^k=\dot\xi_s^k\,ds=e_0^k(s)\,ds$, and setting $dM_s^k:=\sum_{j=1}^d\sqrt{\Xi(\xi_s,\dot\xi_s)}\,e_j^k(s)\,dw_s^j$,

$$d\dot\xi_s^k=dM_s^k-\Gamma_{i\ell}^k(\xi_s)\,\dot\xi_s^i\,\dot\xi_s^\ell\,ds+\tfrac d2\,\Xi\,(\xi_s,\dot\xi_s)\,\dot\xi_s^k\,ds+\tfrac12\,[\dot\xi_s^k\,\dot\xi_s^\ell-g^{k\ell}(\xi_s)]\,\frac{\partial\Xi}{\partial\dot\xi^\ell}(\xi_s,\dot\xi_s)\,ds\,;$$

$$de_j^k(s)=\sqrt{\Xi(\xi_s,\dot\xi_s)}\,\dot\xi_s^k\,dw_s^j-\Gamma_{i\ell}^k(\xi_s)\,e_j^\ell(s)\,\dot\xi_s^i\,ds+\tfrac12\,\Xi\,(\xi_s,\dot\xi_s)\,e_j^k(s)\,ds+\tfrac12\,V_j\,\Xi\,(\xi_s,\dot\xi_s)\,\dot\xi_s^k\,ds\,.$$

Note that we used the formula $\sum_{j=1}^d e_j^k(s)e_j^\ell(s)=\dot\xi_s^k\,\dot\xi_s^\ell-g^{k\ell}(\xi_s)$, which expresses that $\Phi\in G(\mathcal M)$. Using again this formula, we get the quadratic covariation matrix of the martingale term (dM_s), displayed in the above proposition, which shows that $(\pi_1(\Phi_s))$ is indeed a diffusion, and proves Corollary 3.2.2.

On the other hand, comparing the above equations with Equations (23) and (24), which correspond to $\Xi\equiv\varrho^2$, we get precisely the wanted form (28) for the generator of (Φ_s). Then, since $\nabla^v\,\Xi\,\nabla^v=\Xi\,\Delta^v+(\nabla^v\,\Xi)\,\nabla^v$, comparing the above equation (30) for $\dot\xi_s^k$ with Equation (25) (for which $\Xi\equiv\varrho^2$), we see that establishing Formula (29) giving the projected generator $\mathcal H^1_\Xi$ reduces now to proving that $(\nabla^v\,\Xi)\,\nabla^v\equiv[\dot\xi_s^k\,\dot\xi_s^\ell-g^{k\ell}]\,\dfrac{\partial\Xi}{\partial\dot\xi^\ell}\,\dfrac{\partial}{\partial\dot\xi^k}\,.$

Now this becomes clear, noting that $\nabla^v_j = e^k_j \dfrac{\partial}{\partial \dot\xi^k}$, for each $1 \le j \le d$.

Finally, the assertions relative to the Liouville measure are direct consequences of the fact that the vectors H_0 and V_j are antisymmetric with respect to the Liouville measure of $G(\mathcal{M})$. (The invariance with respect to H_0 , i.e. to the geodesic flow, is proved in the same way as in the Riemannian case, and the invariance with respect to V_j is straightforward.) ◇

Remark 3.2.3 (i) The vertical terms could be seen as an effect of the matter or the radiation present in the space-time $\mathcal{M}$. The Ξ-diffusion (Φ_s) reduces to the geodesic flow in the regions of the space where Ξ vanishes, which happens in particular for empty space-times $\mathcal{M}$ in the cases $\Xi = -\varrho^2 R(\xi)$, or $\Xi = \varrho^2 \mathcal{E}(\xi,\dot\xi)$, or also $\Xi = -\varrho^2 R(\xi)\, e^{\kappa\, \mathcal{E}(\xi,\dot\xi)/R(\xi)}$ (for positive constant κ) for example.

(ii) As well as for the basic relativistic diffusion, the law of the Ξ-relativistic diffusion is covariant with any isometry of $(\mathcal{M}, g)$. The basic relativistic diffusion corresponds to $\Xi \equiv \varrho^2 > 0$, and the geodesic flow to $\Xi \equiv 0$.

(iii) In [B] is considered a general model for relativistic diffusions, which may be covariant or not. Up to enlarge it slightly, by allowing the "rest frame" (denoted by z in [B]) to have space vectors of non-unit norm, this model includes the generic Ξ-diffusion $\big($compare the above equation (27) to (2.5),(3.3) in [B]$\big)$.

3.3 The R-diffusion

We assume here that the scalar curvature $R = R(\xi)$ is everywhere non-positive on $\mathcal{M}$, which is physically relevant (see [L-L]), and consider the particular case of Section 3.2 corresponding to $\Xi = -\varrho^2 R(\xi)$, with a constant positive parameter ϱ .

In this case, as its central term clearly vanishes, Equation (27) takes on the simple form:

$$d\Phi_s = H_0(\Phi_s)\, ds + \varrho \sum_{j=1}^{d} \sqrt{-R(\Phi_s)}\, V_j(\Phi_s) \circ dw^j_s \,.$$

3.4 The $\mathcal{E}$-diffusion

We assume that the Weak Energy Condition (recall Section 2.2) holds (everywhere on $T^1\mathcal{M}$), which is physically relevant (see [L-L], [H-E]), and consider the particular case of Section 3.2 corresponding to $\Xi = \varrho^2 \mathcal{E} = \varrho^2 \mathcal{E}(\xi,\dot\xi) = \varrho^2\, T_{00}$.

We call energy relativistic diffusion or $\mathcal{E}$-diffusion the $G(\mathcal{M})$-valued diffusion process (Φ_s) we get in this way, as well as its $T^1\mathcal{M}$-valued projection $\pi_1(\Phi_s)$.

The following is easily derived from Lemma 2.2.2 and Formula (10). As a consequence, the central drift term in Equation (27) is a function of the Ricci tensor alone when Ξ is.

Lemma 3.4.1 *We have* $V_j R_i^k = \delta_{0i}\, R_j^k - \eta_{ij}\, R_0^k + \delta_0^k\, R_{ij} - \delta_j^k\, R_{0i}$ *, for* $0 \le i,k \le d$ *and* $1 \le j \le d$ *. In particular,* $V_j R = 0$ *, and* $V_j \mathcal{E} = V_j T_{00} = V_j R_{00} = 2R_{0j}$ *.*

By Lemma 3.4.1, the drift term of Equation (30) which involves the derivatives of Ξ equals here: $\varrho^2 \sum_{j=1}^{d} R_{0j}(\xi_s)\, e_j^k(s)\, ds$. As we have $R_{0j} = \tilde{R}_{mn}\, e_0^m\, e_j^n$ by (16), we get the alternative expression: $\varrho^2 \sum_{j=1}^{d} R_{0j}(\xi_s)\, e_j^k(s)\, ds = \varrho^2\, \tilde{R}_{mn}(\xi_s)\, \dot{\xi}^m\, [\dot{\xi}_s^k\, \dot{\xi}_s^n - g^{kn}(\xi_s)]\, ds$.

Another expression is got by using Einstein equation (17), or equivalently, by computing directly from (30) and (12): $[\dot{\xi}^k \dot{\xi}^\ell - g^{k\ell}]\, \dfrac{\partial \mathcal{E}}{\partial \dot{\xi}^\ell} = 2[\dot{\xi}^k \dot{\xi}^\ell - g^{k\ell}]\, \tilde{T}_{\ell m} \dot{\xi}^m = 2[\mathcal{E}\, \dot{\xi} - \tilde{T}\dot{\xi}]^k$, where the notation $(\tilde{T}\dot{\xi})^k \equiv \tilde{T}_m^k \dot{\xi}^m$ has the meaning of a matrix product.

Hence, Formula (30) of Proposition 3.2.2 expressing $d\dot{\xi}_s$ reads here:

$$d\dot{\xi}_s^k = dM_s^k - \Gamma_{ij}^k(\xi_s)\, \dot{\xi}_s^i\, \dot{\xi}_s^j\, ds + \tfrac{\varrho^2 d}{2}\, \mathcal{E}_s\, \dot{\xi}_s^k\, ds + \varrho^2\, \tilde{R}_{mn}(\xi_s)\, \dot{\xi}^m\, [\dot{\xi}_s^k\, \dot{\xi}_s^n - g^{kn}(\xi_s)]\, ds\,, \qquad (31)$$

or equivalently:

$$d\dot{\xi}_s = dM_s - \Gamma_{ij}^{\cdot}(\xi_s)\, \dot{\xi}_s^i\, \dot{\xi}_s^j\, ds + \varrho^2 (\tfrac{d}{2} + 1)\, \mathcal{E}_s\, \dot{\xi}_s\, ds - \varrho^2\, \tilde{T}\dot{\xi}_s\, ds\,. \qquad (32)$$

We can then compute the equation satisfied by the random energy $\mathcal{E}_s$. In particular, the drift term discussed above for $d\dot{\xi}_s^k$ contributes now for:

$$2\varrho^2\, \tilde{T}_{km}\, \dot{\xi}^m [\mathcal{E}\, \dot{\xi} - \tilde{T}\dot{\xi}]^k = 2\varrho^2 (\mathcal{E}\, \tilde{T}_{km}\, \dot{\xi}^m \dot{\xi}^k - \tilde{T}_{km}\, \dot{\xi}^m [\tilde{T}\dot{\xi}]^k) = 2\varrho^2 (\mathcal{E}^2 - [\tilde{T}\dot{\xi}]_k\, [\tilde{T}\dot{\xi}]^k).$$

This leads to the following, to be compared with Corollary 3.1.2.

Remark 3.4.2 *The random energy* $\mathcal{E}_s := \mathcal{E}(\xi_s, \dot{\xi}_s)$ *associated to the* $\mathcal{E}$*-diffusion* (Φ_s) *satisfies the following equation (where* $\nabla_v := v^j \nabla_j$*):*

$$d\mathcal{E}_s = \nabla_{\dot{\xi}_s} \mathcal{E}(\xi_s, \dot{\xi}_s)\, ds + (d+2)\, \varrho^2\, \mathcal{E}_s^2\, ds - 2\varrho^2\, g(\tilde{T}\dot{\xi}_s, \tilde{T}\dot{\xi}_s)\, ds + 2\varrho\, dM_s^{\mathcal{E}}\,,$$

with the quadratic variation of its martingale part $dM_s^{\mathcal{E}}$ *given by:*

$$[d\mathcal{E}_s, d\mathcal{E}_s] = 4\varrho^2\, [dM_s^{\mathcal{E}}, dM_s^{\mathcal{E}}] = 4\varrho^2\, [\mathcal{E}_s^2 - g(\tilde{T}\dot{\xi}_s, \tilde{T}\dot{\xi}_s)]\, \mathcal{E}_s\, ds\,.$$

Note that the diffusion coefficient $[\mathcal{E}_s^2 - g(\tilde{T}\dot{\xi}_s, \tilde{T}\dot{\xi}_s)]$ appearing in the quadratic variation $[d\mathcal{E}_s, d\mathcal{E}_s]$ is necessarily non-negative, which can of course be checked directly.

Remark 3.4.3 *Case of Einstein Lorentz manifolds.*

The Lorentz manifold $\mathcal{M}$ is said to be Einstein if its Ricci tensor is proportional to its metric tensor. Bianchi's contracted identities (see [H-E] or [K-N]), which entail the

conservation equations $\nabla_k \tilde{T}^{jk} = 0$, force the proportionality coefficient $\tilde{p}$ to be constant on $\mathcal{M}$. Hence:

$$\tilde{R}_{\ell m}(\xi) = \tilde{p}\, g_{\ell m}(\xi)\,, \quad \text{for any } \xi \text{ in } \mathcal{M} \text{ and } 0 \le \ell, m \le d\,.$$

Then the scalar curvature is $R(\xi) = (d+1)\tilde{p}$, and by Einstein Equations (17) we have:

$$\tilde{T}_{\ell m}(\xi) = (\Lambda - \tfrac{d-1}{2}\,\tilde{p})\, g_{\ell m}(\xi) \;=:\; -p\, g_{\ell m}(\xi)\,.$$

Hence Equation (19) holds, with $q = 0$: we are in a limiting case of perfect fluid. Moreover, R and $\mathcal{E}$ are constant, so that on an Einstein Lorentz manifold, the R-diffusion and the $\mathcal{E}$-diffusion coincide with the basic relativistic diffusion (of Section 3.1).

4 Warped (or skew) products

Let us consider here a Lorentz manifold $(\mathcal{M}, g)$ having the warped product form: $\mathcal{M} = I \times M$, where I is an open interval of $\mathbb{R}_+$ and (M, h) is a Riemannian manifold, is endowed with the Lorentzian pseudo-norm g given by:

$$ds^2 := dt^2 - \alpha(t)^2\, |dx|_h^2\,, \tag{33}$$

or equivalently

$$g_{0k} := \delta_{0k} \quad \text{and} \quad g_{ij} := -\alpha(t)^2\, h_{ij}(x)\,, \;\; \text{for } 0 \le k \le d\,,\; 1 \le i, j \le d\,. \tag{34}$$

Here $\xi \equiv (t, x) \in I \times M$ denotes the generic point of $\mathcal{M}$, and the expansion factor α is a positive C^2 function on I. The so-called Hubble function is:

$$H(t) := \alpha'(t)/\alpha(t)\,. \tag{35}$$

This structure is considered in [B-E], which contains most of the following proposition.

Proposition 4.1 *Consider a Lorentz manifold $(\mathcal{M}, g)$ having the warped product form.*
(i) Its curvature operator $\mathcal{R}$ is given by: for $u, v, w, a \in C^2(I)$ and $X, Y, Z, A \in \Gamma(TM)$,

$$\Big\langle \mathcal{R}\left((u\partial_t + X) \wedge (v\partial_t + Y)\right), (a\partial_t + A) \wedge (w\partial_t + Z) \Big\rangle_g = -\alpha^2 \left\langle \mathcal{K}\,(X \wedge Y)\,,\, A \wedge Z \right\rangle$$
$$+\, \alpha\alpha''\, h(u\,Y - v\,X, a\,Z - w\,A) + (\alpha\alpha')^2 \left[h(X, Z)\, h(Y, A) - h(X, A)\, h(Y, Z)\right], \tag{36}$$

$\mathcal{K}$ denoting the curvature operator of (M, h).

(ii) Denoting by Ric *the Ricci operator of (M, h) and by $\langle \cdot,\cdot \rangle$ the standard canonical inner product of $\mathbb{R}^d$, we have:*

$$\Big\langle \mathrm{Ricci}(v\partial_t + Y), w\partial_t + Z \Big\rangle_\eta = \langle \mathrm{Ric}(Y), Z\rangle + \left[(d-1)\,|\alpha'|^2 + \alpha\alpha''\right] h(Y, Z) - d\,\frac{\alpha''}{\alpha}\, vw. \tag{37}$$

(iii) If ∇^g and ∇^h denote the Levi-Civita connections of $(\mathcal{M}, g)$ and (M, h) respectively:

$$\nabla^g_{(u\partial_t + X)}(v\partial_t + Y) = \left[uv' + \alpha(t)\alpha'(t)\, h(X, Y)\right] \partial_t + \nabla^h_X Y + H(t)\,(u\,Y + v\,X)\,. \tag{38}$$

Remark 4.2 *In any warped local chart* (t, x^j) *: for* $1 \le m, n, p, q \le d$ *(*K_{mnpq} *denoting the curvature tensor of* (M, h)*, and* K_{mp} *its Ricci tensor) and* $0 \le k \le d$ *, we have*

$$\Gamma^{\cdot}_{00}(g) = 0\ ; \quad \Gamma^p_{0n}(g) = H\,\delta^p_n\ ; \quad \Gamma^0_{mn}(g) = \alpha\alpha'\,h_{mn}\ ; \quad \Gamma^p_{mn}(g) = \Gamma^p_{mn}(h)\,, \tag{39}$$

$$\widetilde{\mathcal{R}}_{0nkq} = \delta_{0k}\,\alpha(t)\alpha''(t)\,h_{nq}\ ; \quad \widetilde{\mathcal{R}}_{mnpq} = (\alpha\alpha')^2(t)\,[h_{mq}\,h_{np} - h_{mp}\,h_{nq}] - \alpha^2(t)\widetilde{K}_{mnpq}\,, \tag{40}$$

$$\tilde{R}_{0k} = -\,\delta_{0k}\,d \times \alpha''(t)/\alpha(t)\ ; \quad \tilde{R}_{mp} = \tilde{K}_{mp} + [(d-1)\,|\alpha'|^2 + \alpha\alpha'']\,h_{mp}\,. \tag{41}$$

Let us outline the proof for the convenience of the reader.

Proof We get (38), and then (39), by using Koszul formula:

$$2\,h(\nabla^h_X Y, Z) = Xh(Y,Z) + Yh(X,Z) - Zh(X,Y) + h([X,Y],Z) - h([X,Z],Y) + h([Z,Y],X),$$

for both ∇^h and ∇^g. Hence,

$$\nabla^g_{(u\partial_t+X)}\nabla^g_{(v\partial_t+Y)}(w\partial_t + Z) = \nabla^g_{(u\partial_t+X)}\Big([vw' + \alpha\alpha'\,h(Y,Z)]\partial_t + \nabla^h_Y Z + H\,[vZ + wY]\Big)$$

Therefore

$$\Big[\nabla^g_{(u\partial_t+X)}, \nabla^g_{(v\partial_t+Y)}\Big](w\partial_t + Z)$$

And since

$$\nabla^g_{[(u\partial_t+X),(v\partial_t+Y)]}(w\partial_t + Z) = \nabla^g_{([uv'-u'v]\,\partial_t+[X,Y])}(w\partial_t + Z)$$

$$= \Big([uv' - u'v]w' + \alpha\alpha'\,h([X,Y],Z)\Big)\partial_t + \nabla^h_{[X,Y]}Z + H\,([uv' - u'v]\,Z + w\,[X,Y])\,,$$

we get

$$\Big(\Big[\nabla^g_{(u\partial_t+X)}, \nabla^g_{(v\partial_t+Y)}\Big] - \nabla^g_{[(u\partial_t+X),(v\partial_t+Y)]}\Big)(w\partial_t + Z)$$

$$= \alpha\alpha'' h(uY - vX, Z)\partial_t + ([\nabla^h_X, \nabla^h_Y] - \nabla^h_{[X,Y]})Z + \frac{\alpha''}{\alpha}w(uY - vX) + \alpha'^2[h(Y,Z)X - h(X,Z)Y].$$

By Formula (9), this entails Formula (36), which is equivalent to (40), by (15).

Then, denoting by $(e_1, \dots, e_d)$ an orthonormal basis of (M, h):

which yields (37); which is in turn equivalent to (41), by (16). ⋄

Corollary 4.3 *A Lorentz manifold* $(\mathcal{M}, g)$ *having the warped product form is of perfect fluid type (recall Section 2.4) if and only if the Ricci operator of its Riemannian factor* M *is conformal to the identical map:* $\mathrm{Ric} = \Omega \times \mathrm{Id}$ *, for some* $\Omega \in C^0(M)$*.*

If this holds, we must have: $U = \partial_t$ *,* $\tilde{p} + q = -d\,(\alpha''/\alpha)$*, and* $q = \Omega\,\alpha^{-2} - (d-1)H'$*.*

Proof By (21) and (37), this happens if and only if for any $v \in C^0(I), Y \in \Gamma(TM)$:

$\langle \mathrm{Ric}(Y), Y\rangle + [(d-1)\alpha'^{\,2} + \alpha\alpha'']h(Y,Y) - d(\alpha''/\alpha)v^2 = q\, g(U, v\partial_t + Y)^2 + \tilde{p}\,[v^2 - \alpha^2 h(Y,Y)]$,

which (seen as a polynomial in v) forces $U = \partial_t$, and then splits into $\tilde{p} + q = -d\,(\alpha''/\alpha)$ and $\langle \mathrm{Ric}(Y), Y\rangle = -[(d-1)\alpha'^{\,2} + \alpha\alpha'' + \tilde{p}\,\alpha^2]\, h(Y,Y)$. This latter equation is equivalent to $\tilde{p} = -[(d-1)H^2 + (\alpha''/\alpha) + \Omega\,\alpha^{-2}]$, and then, using $\tilde{p} + q = -d\,(\alpha''/\alpha)$, to $q = \Omega\,\alpha^{-2} - (d-1)H'$. $\diamond$

Corollary 4.4 *Consider a Lorentz manifold $(\mathcal{M}, g)$ having the warped product form. The energy (12) at $\dot{\xi} \equiv (\dot{t}, \dot{x}) \in T^1_\xi\mathcal{M} \equiv T^1_{(t,x)}(I \times M)$ equals:*

$$\mathcal{E}(\xi, \dot{\xi}) = \langle \mathrm{Ric}(\dot{x}), \dot{x}\rangle - (d-1)\, H'(t)\, (\dot{t}^2 - 1) + \tfrac{1}{2}\, d\,(d-1)\, H^2(t) + \frac{R^M}{2\,\alpha^2(t)}\,. \tag{42}$$

Then the weak energy condition is equivalent to the following lower bounds for the Ricci operator and the scalar curvature R^M of the Riemannian factor (M, h):

$$\inf_{w \in TM} \frac{\langle \mathrm{Ric}(w), w\rangle}{h(w,w)} \geq (d-1)\, \sup_I\, \{\alpha\,\alpha'' - \alpha'^{\,2}\}\,; \quad R^M \geq -d\,(d-1)\, \inf_I\, \alpha'^{\,2}\,. \tag{43}$$

And the scalar curvature R of $(\mathcal{M}, g)$ equals:

$$R = -\alpha^{-2} R^M - d\Big[(d-1)\, |\tfrac{\alpha'}{\alpha}|^2 + 2\, \tfrac{\alpha''}{\alpha}\Big], \tag{44}$$

so that its non-positivity is equivalent to the lower bound on the scalar curvature R^M:

$$R^M \geq -d \times \inf_I\, \{(d-1)\, \alpha'^{\,2} + 2\, \alpha\, \alpha''\}\,. \tag{45}$$

Proof From (37), we compute the scalar curvature of $\mathcal{M}$:

$$R = \langle \mathrm{Ricci}(\partial_t), \partial_t\rangle_\eta - \alpha^{-2} \sum_{j=1}^{d} \langle \mathrm{Ricci}(e_j), e_j\rangle_\eta = -\alpha^{-2} R^M - d\Big[(d-1)\, |\tfrac{\alpha'}{\alpha}|^2 + 2\, \tfrac{\alpha''}{\alpha}\Big].$$

On the other hand, by (37) we have at $\dot{\xi} \equiv (\dot{t}, \dot{x}) \in T^1_\xi\mathcal{M} \equiv T^1_{(t,x)}(I \times M)$:

$$\begin{aligned}\langle \mathrm{Ricci}(\dot{\xi}), \dot{\xi}\rangle_\eta &= \langle \mathrm{Ric}(\dot{x}), \dot{x}\rangle + [(d-1)\, |\alpha'|^2 + \alpha\alpha'']\, \tfrac{\dot{t}^2 - 1}{\alpha^2} - d\, \tfrac{\alpha''}{\alpha}\, \dot{t}^2 \\ &= \langle \mathrm{Ric}(\dot{x}), \dot{x}\rangle - (d-1)\, H'(t)\, (\dot{t}^2 - 1) - d\, \tfrac{\alpha''}{\alpha}(t)\,.\end{aligned}$$

Thence Formula (42). Then, the weak energy condition holds if and only if for any $t \in I$ and $w \in TM$:

$$\langle \mathrm{Ric}(w), w\rangle \geq (d-1)\, \alpha^2(t) H'(t)\, h(w,w) - \tfrac{1}{2}\, d\,(d-1)\, H^2(t) - \frac{R^M}{2\,\alpha^2(t)}\,.$$

By homogeneity with respect to w, this can be split into the following lower bound for the Ricci operator of the Riemannian factor M:

$$\inf_{w\in TM} \frac{\langle \mathrm{Ric}(w), w\rangle}{h(w,w)} \geq (d-1) \sup_I \{\alpha\, \alpha'' - \alpha'^{\,2}\},$$

together with the condition particularised to $w = 0$, which yields the following lower bound for the scalar curvature of the Riemannian factor M:

$$d(d-1)\, H^2(t) + \alpha(t)^{-2} R^M \geq 0\,, \quad \text{or} \qquad R^M \geq -d\,(d-1) \inf_I \alpha'^{\,2}. \ \diamond$$

5 Example of Robertson-Walker (R-W) manifolds

These important manifolds are particular cases of warped product: they can be written $\mathcal{M} = I \times M$, where I is an open interval of $\mathbb{R}_+$ and $M \in \{\mathbb{S}^3, \mathbb{R}^3, \mathbb{H}^3\}$, with spherical coordinates $\xi \equiv (t, r, \varphi, \psi)$ (which are global in the case of $\mathbb{R}^3, \mathbb{H}^3$, and are defined separately on two hemispheres in the case of $\mathbb{S}^3$), and are endowed with the pseudo-norm:

$$g(\dot\xi, \dot\xi) := \dot t^2 - \alpha(t)^2 \left(\frac{\dot r^2}{1-kr^2} + r^2 \dot\varphi^2 + r^2 \sin^2\!\varphi\ \dot\psi^2 \right), \tag{46}$$

where the constant scalar spatial curvature k belongs to $\{-1, 0, 1\}$ (note that $r \in [0,1]$ for $k = 1$ and $r \in \mathbb{R}_+$ for $k = 0, -1$), and the expansion factor α is as in the previous section 4. Note that we have necessarily $\dot t \geq 1$ everywhere on $T^1\mathcal{M}$.

By (36), we have the curvature operator given by:

$$\Big\langle \mathcal{R}\left((u\partial_t + X) \wedge (v\partial_t + Y)\right), (a\partial_t + A) \wedge (w\partial_t + Z) \Big\rangle_\eta$$

$$= \alpha\alpha''\, h(uY - vX, aZ - wA) - \alpha^2(\alpha'^{\,2} + k)\, [h(X,A)h(Y,Z) - h(X,Z)h(Y,A)]\,.$$

By (37), the Ricci tensor $(\!(\tilde R_{\ell m})\!)$ is diagonal, with diagonal entries:

$$\Big(-3\,\frac{\alpha''(t)}{\alpha(t)}\,, \frac{A(t)}{1-kr^2}\,, A(t)\, r^2, A(t)\, r^2 \sin^2\!\varphi\Big), \quad \text{where } \ A(t) := \alpha(t)\,\alpha''(t) + 2\,\alpha'(t)^2 + 2k\,,$$

and the scalar curvature is $\quad R = -6\,[\alpha(t)\,\alpha''(t) + \alpha'(t)^2 + k]\,\alpha(t)^{-2}$.

The Einstein energy-momentum tensor $\tilde R_{\ell m} - \frac{1}{2} R\, g_{\ell m} = \tilde T_{\ell m}$ is diagonal as well, with diagonal entries:

$$\Big(3\,\frac{\alpha'(t)^2 + k}{\alpha(t)^2}\,, \frac{-\tilde A(t)}{1-kr^2}\,, -\tilde A(t)\, r^2,\ -\tilde A(t)\, r^2 \sin^2\!\varphi\Big), \ \text{ with } \ \tilde A(t) := 2\alpha(t)\alpha''(t) + \alpha'(t)^2 + k\,.$$

Hence, we have

$$\tilde T_{\ell m} - \alpha(t)^{-2} \tilde A(t)\, g_{\ell m} = 2\,[k\, \alpha(t)^{-2} - H'(t)]\, 1_{\{\ell = m = 0\}}\,.$$

Thus, in accordance with Corollary 4.3, we have here an example of perfect fluid : Equation (19) holds, with

$$U_j \equiv \delta_j^0\,, \quad -p(\xi) = k\,\alpha(t)^{-2} + 2H'(t) + 3H^2(t)\,, \quad q(\xi) = 2\,[k\,\alpha(t)^{-2} - H'(t)]\,, \tag{47}$$

$$\tilde{p}(\xi) = -2\,[2k\,\alpha(t)^{-2} + H'(t) + 3H^2(t)]/(d-1)\,.$$

Note that

$$\mathcal{A}_s = U_i(\xi_s)\dot{\xi}_s^i = \dot{t}_s \quad \text{and} \quad \mathcal{E}_s = 2\,[k\,\alpha(t_s)^{-2} - H'(t_s)]\,\dot{t}_s^2 - p(\xi_s)\,. \tag{48}$$

By Corollary 4.4 (or by Remark 2.4.1(ii) as well), the weak energy condition is equivalent to : $\alpha'^{\,2} + k \,\geq\, (\alpha\,\alpha'')^+$.

We shall consider only eternal Robertson-Walker space-times, which have their future-directed half-geodesics complete. This amounts to $I = \mathbb{R}_+^*$, together with $\displaystyle\int^\infty \frac{\alpha}{\sqrt{1+\alpha^2}} = \infty$. In the case of the basic relativistic diffusion (solving Equation (23) in such Robertson-Walker model), we have in particular :

$$d\dot{t}_s = \varrho\sqrt{\dot{t}_s^2 - 1}\,dw_s + \tfrac{3\varrho^2}{2}\,\dot{t}_s\,ds - H(t_s)[\dot{t}_s^2 - 1]\,ds\,. \tag{49}$$

5.1 Ξ-relativistic diffusions in an Einstein-de Sitter-like manifold

We consider henceforth the particular case $I =]0,\infty[$, $k=0$, and $\alpha(t) = t^c$, with exponent $c > 0$. Note that such expansion functions α can be obtained by solving a proportionality relation between p and q (see [H-E] or [L-L]).

Thus $q = 2c\,t^{-2}$, $p = (2-3c)c\,t^{-2}$, $R = -6\,c\,(2c-1)\,t^{-2}$, $\mathcal{E} = c\,t^{-2}\,(2\,\dot{t}^2 + 3c - 2)$.

Note that the weak energy condition holds. The scalar curvature is non-positive if and only if $c \geq 1/2$, and the pressure p is non-negative if and only if $c \leq 2/3$.

Note that the particular case $c = \frac{2}{3}$ corresponds to a vanishing pressure p, and is precisely known as that of Einstein-de Sitter universe (see for example [H-E]). And the analysis of [L-L] shows up precisely both limiting cases $c = \frac{2}{3}$ and $c = \frac{1}{2}$.

5.1.1 Basic relativistic diffusion in an Einstein-de Sitter-like manifold

In order to compare with the other relativistic diffusions, we mention first for the basic relativistic diffusion (of Section 3.1), the stochastic differential equations satisfied by the main coordinates $\dot{t}_s$ and $\dot{r}_s$, appearing in the 4-dimensional sub-diffusion $(t_s, \dot{t}_s, r_s, \dot{r}_s)$. By (49), we have, for independent standard real Brownian motions $w, \tilde{w}$:

$$d\dot{t}_s \;=\; \varrho\sqrt{\dot{t}_s^2 - 1}\,dw_s + \tfrac{3\varrho^2}{2}\,\dot{t}_s\,ds - \frac{c}{t_s}\,(\dot{t}_s^2 - 1)\,ds\,; \tag{50}$$

$$d\dot r_s = \frac{\varrho\,\dot t_s\,\dot r_s}{\sqrt{\dot t_s^2-1}}\,dw_s + \varrho\sqrt{\frac{1}{t_s^{2c}} - \frac{\dot r_s^2}{\dot t_s^2-1}}\,d\tilde w_s + \tfrac{3\varrho^2}{2}\,\dot r_s\,ds + \left[\frac{\dot t_s^2-1}{t_s^{2c}} - \dot r_s^2\right]\frac{ds}{r_s} - \frac{2c}{t_s}\,\dot t_s\,\dot r_s\,ds\,. \quad (51)$$

Almost surely (see [A]), $\lim\limits_{s\to\infty} \dot t_s = \infty$, and $x_s/r_s \sim \dot x_s/|\dot x_s|$ converges in $\mathbb{S}^2$.

5.1.2 R-diffusion in an Einstein-de Sitter-like manifold

With the above, Section 3.3 reads here, for the R-relativistic diffusion, when $c \geq 1/2$:

$$d\dot\xi_s = \varrho\,dM_s + 9c\,(2c-1)\varrho^2\,t_s^{-2}\,\dot\xi_s\,ds - \Gamma_{ij}^{\cdot}(\xi_s)\,\dot\xi_s^i\,\dot\xi_s^j\,ds\,, \qquad (52)$$

with the quadratic covariation matrix of the martingale part dM_s given by:

$$\varrho^{-2}\,[d\dot\xi_s^k, d\dot\xi_s^\ell] = 6c\,(2c-1)\,[\dot\xi_s^k\,\dot\xi_s^\ell - g^{k\ell}(\xi_s)]\,t_s^{-2}\,ds\,, \quad \text{for } 0\leq k,\ell\leq d\,.$$

In particular, we have for independent standard real Brownian motions $w, \tilde w$:

$$d\dot t_s \;=\; \frac{\varrho}{t_s}\sqrt{6c\,(2c-1)(\dot t_s^2-1)}\,dw_s + \frac{9\varrho^2 c\,(2c-1)}{t_s^2}\,\dot t_s\,ds - \frac{c}{t_s}(\dot t_s^2-1)\,ds\,; \qquad (53)$$

$$\begin{aligned} d\dot r_s \;=\; & \frac{\varrho\sqrt{6c\,(2c-1)}}{t_s}\left[\frac{\dot t_s\,\dot r_s}{\sqrt{\dot t_s^2-1}}\,dw_s + \sqrt{\frac{1}{t_s^{2c}} - \frac{\dot r_s^2}{\dot t_s^2-1}}\,d\tilde w_s\right] \qquad (54)\\ & + \frac{9\varrho^2 c\,(2c-1)}{t_s^2}\,\dot r_s\,ds + \left[\frac{\dot t_s^2-1}{t_s^{2c}} - \dot r_s^2\right]\frac{ds}{r_s} - \frac{2c}{t_s}\,\dot t_s\,\dot r_s\,ds\,. \end{aligned}$$

As the scalar curvature $R_s = 6c\,(1-2c)/t_s^2$ vanishes asymptotically, we expect that almost surely the R-diffusion behaves eventually as a timelike geodesic, and in particular that $\lim\limits_{s\to\infty} \dot t_s = 1$.

5.1.3 $\mathcal{E}$-diffusion in an Einstein-de Sitter-like manifold

Similarly, using (32), (47), (48), we have here $\mathcal{E}\,\dot\xi - \tilde T\dot\xi = 2(0-H')(\dot t^2\dot\xi - \dot t\,U)$, so that Section 3.4 reads here, for the $\mathcal{E}$-diffusion:

$$d\dot\xi_s = \varrho\,dM_s + \tfrac{3\varrho^2 c}{2}\,t_s^{-2}\,(2\,\dot t_s^2 + 3c - 2)\,\dot\xi_s\,ds + 2\varrho^2 c\,t_s^{-2}(\dot t_s\,\dot\xi_s - U_s)\,\dot t_s\,ds - \Gamma_{ij}^{\cdot}(\xi_s)\,\dot\xi_s^i\,\dot\xi_s^j\,ds\,, \quad (55)$$

with the quadratic covariation matrix of the martingale part dM_s given by:

$$\varrho^{-2}\,[d\dot\xi_s^k, d\dot\xi_s^\ell] = c\,[\dot\xi_s^k\,\dot\xi_s^\ell - g^{k\ell}(\xi_s)]\,(2\,\dot t_s^2 + 3c - 2)\,t_s^{-2}\,ds\,, \quad \text{for } 0\leq k,\ell\leq d\,.$$

In particular, we have for some standard real Brownian motion w:

$$d\dot t_s = \frac{\varrho\sqrt{c}}{t_s}\sqrt{(2\,\dot t_s^2 - 2 + 3c)(\dot t_s^2-1)}\,dw_s + c\left[5\varrho^2(\dot t_s^2 - 1 + \tfrac{9c}{10})\,\frac{\dot t_s}{t_s^2} - \frac{\dot t_s^2-1}{t_s}\right]ds\,; \qquad (56)$$

$$d\dot r_s = \frac{\varrho\sqrt{c}}{t_s}\sqrt{2\,\dot t_s^2 - 2 + 3c}\,\left[\frac{\dot t_s\,\dot r_s}{\sqrt{\dot t_s^2 - 1}}\,dw_s + \sqrt{\frac{1}{t_s^{2c}} - \frac{\dot r_s^2}{\dot t_s^2 - 1}}\,d\tilde w_s\right] \qquad (57)$$

$$+\,\varrho^2 c\,(5\,\dot t_s^2 - 3 + \tfrac{9c}{2})\,\frac{\dot r_s}{t_s^2}\,ds - \frac{2c}{t_s}\,\dot t_s\,\dot r_s\,ds + \left[\frac{\dot t_s^2 - 1}{t_s^{2c}} - \dot r_s^2\right]\frac{ds}{r_s}\,.$$

Remark 5.1.4 *Comparison of Ξ-diffusions in an E.-d.S.-like manifold*

Along the preceding sections 5.1.1, 5.1.2, 5.1.3, we specified to an Einstein-de Sitter-like manifold the various Ξ-diffusions we considered successively in Sections 3.1, 3.3, 3.4. Restricting to the only equation relating to the hyperbolic angle $\mathcal{A}_s = \dot t_s$, or in other words, to the simplest sub-diffusion $(t_s, \dot t_s)$, this yields Equations (50), (53), (56) respectively. We observe that even in this simple case, all these covariant relativistic diffusions differ notably, having pairwise distinct minimal sub-diffusions (with 3 non-proportional diffusion factors).

5.2 Asymptotic behavior of the R-diffusion in an Einstein-de Sitter-like manifold

We present here the asymptotic study of the R-diffusion of an Einstein-de Sitter-like manifold (recall Sections 5.1, 5.1.2). We will focus our attention on the simplest sub-diffusion $(t_s, \dot t_s)$, and on the space component $x_s \in \mathbb{R}^3$. Recall from (48) that $\dot t_s = \mathcal{A}_s$ equals the hyperbolic angle, measuring the gap between the ambient fluid and the velocity of the diffusing particle. Recall also that, by the unit pseudo-norm relation, $\dot t_s$ controls the behavior of the whole velocity $\dot\xi_s$. We get as a consequence the asymptotic behavior of the energy $\mathcal{E}_s$. As quoted in Section 5.1.2, we must have here $c \geq \frac12$.

Note that for $c = \frac12$, the scalar curvature vanishes, and the R-diffusion reduces to the geodesic flow, whose equations are easily solved and whose time coordinate satisfies (for constants a and s_0):

$$s - s_0 = \sqrt{t_s\,(t_s + a^2)} - a^2\log[\sqrt{t_s} + \sqrt{t_s + a^2}\,]\,, \quad \text{whence} \quad t_s \sim s\,.$$

The proofs of this section (and of the following one) will use several times the elementary fact that almost surely a continuous local martingale cannot go to infinity.

The following confirms a conjecture stated at the end of Section 5.1.2.

Proposition 5.2.1 *The process $\dot t_s$ goes almost surely to 1, and $\mathcal{E}_s \to 0$, as $s \to \infty$.*

<u>Proof</u> By Equation (53),

$$\log\frac{\dot t_s}{\dot t_1} - 3\varrho^2 c(2c-1)\int_1^s (2 + \dot t_\tau^{-2})\frac{d\tau}{t_\tau^2} + c\int_1^s (1 - \dot t_\tau^{-2})\,\frac{\dot t_\tau}{t_\tau}\,d\tau$$

is a continuous martingale with quadratic variation $6\varrho^2 c\,(2c-1)\int_1^s (1-\dot t_\tau^{-2})\frac{d\tau}{t_\tau^2}\cdot$ Hence, since $\dot t_\tau \geq 1$ and therefore $t_\tau \geq \tau$, the non-negative process

$$\log \dot t_s + c\int_1^s (1-\dot t_\tau^{-2})\,\frac{\dot t_\tau}{t_\tau}\,d\tau$$

converges almost surely as $s\to\infty$. This forces the almost sure convergence of the integral: $\int_1^\infty (1-\dot t_\tau^{-2})\,\frac{\dot t_\tau}{t_\tau}\,d\tau<\infty$, and of $\dot t_s$, towards some $\dot t_\infty\in[1,\infty[$. This implies in turn $t_\tau=\mathcal{O}(\tau)$, hence $\int_1^\infty (1-\dot t_\tau^{-2})\,\frac{d\tau}{\tau}<\infty$, whence finally $\dot t_\infty=1$. $\diamond$

Consider now the functional $a:=t^c\sqrt{\dot t^2-1}$, which is constant along any geodesic.

Lemma 5.2.2 *For $c>\frac12$, the process $a_s:=t_s^c\sqrt{\dot t_s^2-1}$ goes almost surely to infinity, and cannot vanish. Moreover, for any $\varepsilon>0$ we have almost surely: $\int_1^\infty t_s^{2c-2}\,\frac{ds}{a_s^{2+\varepsilon}}<\infty$.*

Proof We get from Equation (53):

$$da_s = \frac{\varrho}{t_s}\sqrt{6c\,(2c-1)(a_s^2+t_s^{2c})}\,dw_s + 3\varrho^2 c\,(2c-1)\,\frac{3\,a_s^2+2\,t_s^{2c}}{t_s^2\,a_s}\,ds\,,$$

and then for any $\varepsilon\in]0,1]$ and for some continuous local martingale M:

$$0\leq a_s^{-\varepsilon} = a_1^{-\varepsilon} - M_s - 3\varepsilon\varrho^2 c\,(2c-1)\int_1^s \frac{[2-\varepsilon]\,a_\tau^2+[1-\varepsilon]\,t_\tau^{2c}}{t_\tau^2\,a_\tau^{2+\varepsilon}}\,d\tau.$$

The signs in this last formula, and the fact that almost surely a continuous local martingale cannot go to infinity, imply the convergence of the last integral and of the martingale term M_s, entailing the almost sure existence of a finite limit a_∞^{-1}, hence of $a_\infty\in]0,\infty]$, and the almost sure convergence of the integral $\int_1^\infty \frac{d\tau}{a_\tau^{2+\varepsilon}\,t_\tau^{2-2c}}<\infty$. Now, by Proposition 5.2.1, this implies $\int_1^\infty \frac{d\tau}{a_\tau^{2+\varepsilon}\,\tau}\leq\int_1^\infty \frac{d\tau}{a_\tau^{2+\varepsilon}\,\tau^{2-2c}}<\infty$, hence $a_\infty=\infty$. Finally, the equation for $a_s^{-\varepsilon}$ forbids also the existence of a finite zero s_0 for a_s. Indeed, $s\nearrow s_0$ would force the martingale term of this equation to go to $-\infty$, which is impossible. $\diamond$

The following reveals the asymptotic behavior of the space component (x_s) for $c>\frac12$.

Proposition 5.2.3 *For $c>\frac12$, the space component converges almost surely (as $s\to\infty$):*

$$x_s\,\to\,x_\infty\in\mathbb{R}^3.$$

Proof $\quad (i)$ Let us consider the non-negative process $u_s := t_s(\dot t_s^2 - 1)$, which is constant for $c = \frac{1}{2}$, and cannot vanish for $c > \frac{1}{2}$, by Lemma 5.2.2. By Equation (53),

$$u_s + (2c-1)\int_1^s \frac{u_\tau\, \dot t_\tau}{t_\tau}\, d\tau \,-\, 6\varrho^2 c\,(2c-1)\int_1^s [4\,\dot t_\tau^2 - 1]\,\frac{d\tau}{t_\tau}$$

is a continuous local martingale. Then, for any $\varepsilon > 0$,

$$\frac{u_s}{t_s^\varepsilon} - 6\varrho^2 c\,[2c-1]\int_1^s \frac{[4\dot t_\tau^2 - 1]\,d\tau}{t_\tau^{1+\varepsilon}} + [2c-1+\varepsilon]\int_1^s \frac{u_\tau\, dt_\tau}{t_\tau^{1+\varepsilon}}$$

is a continuous local martingale. By Proposition 5.2.1, the central term converges almost surely. This implies that $\displaystyle\int_1^\infty \frac{u_\tau\, dt_\tau}{t_\tau^{1+\varepsilon}} < \infty$ and that $\dfrac{u_s}{t_s^\varepsilon}$ converges, almost surely.

(ii) By the unit pseudo-norm relation, we have $t_s^{2c}\,|\dot x_s|^2 = \dot t_s^2 - 1 = u_s/t_s$. Let us apply (i) above with $\varepsilon := c - \frac{1}{2}$, to get:

$$\left[\int_1^\infty |\dot x_s|\, ds\right]^2 \le \int_1^\infty t_s^{\varepsilon-2c}\, ds \times \int_1^\infty t_s^{2c-\varepsilon}\,|\dot x_s|^2\, ds \,\le\, \tfrac{2}{2c-1}\int_1^\infty \frac{u_s}{t_s^{1+\varepsilon}}\, ds \,<\infty\,.$$

This proves that $x_s = x_1 + \displaystyle\int_1^s \dot x_\tau\, d\tau \to x_1 + \int_1^\infty \dot x_s\, ds \in \mathbb{R}^3$, almost surely as $s\to\infty$. $\diamond$

In the case $c = \frac{1}{2}$ of the R-diffusion being the geodesic flow, we have

$$r_s = \sqrt{b^2/a^2 + (a+o(1))\log s} \,\sim\, \sqrt{a\log s} \qquad \text{as } s\to\infty\,,$$

which shows that Proposition 5.2.3 does not hold for the limiting case $c = \frac{1}{2}$.

To compare the R-diffusion with geodesics, note that (as is easily seen; see for example [A]) along any timelike geodesic, we have $x_s = x_1 + \dfrac{\dot x_1}{|\dot x_1|}\displaystyle\int_1^s \frac{a\, d\tau}{t_\tau^{2c}}$ $\big($and $\frac{\dot x_s}{|\dot x_s|} = \frac{\dot x_1}{|\dot x_1|}\big)$, which converges precisely for $c > \frac{1}{2}$; and along any lightlike geodesic, we have $x_s = x_1 + \dfrac{\dot x_1}{|\dot x_1|}\displaystyle\int_{t_1}^{t_s} \frac{d\tau}{\tau^c} \sim V\times s^{\frac{1-c}{1+c}}$ $\big($and $\frac{\dot x_s}{|\dot x_s|} = \frac{\dot x_1}{|\dot x_1|}\big)$, which converges only for $c > 1$.

On the other hand, for $c \le 1$, the behavior of the basic relativistic diffusion proves to satisfy (see [A]): $\quad r_s \underset{s\to\infty}{\sim} \displaystyle\int_1^s \frac{a_\tau\, d\tau}{t_\tau^{2c}} \longrightarrow \infty$ (exponentially fast, at least for $c < 1$).

Hence, the R-diffusion behaves asymptotically more like a (timelike) geodesic than like the basic relativistic diffusion. However, owing to Lemma 5.2.2, the asymptotic behavior of the R-diffusion seems to be somehow intermediate between those of the geodesic flow and of the basic relativistic diffusion.

5.3 Asymptotic energy of the $\mathcal{E}$-diffusion in an E.-d.S. manifold

We consider here the case of Section 5.1.3, dealing with the energy diffusion in an Einstein-de Sitter-like manifold, and more precisely, with its absolute-time minimal sub-diffusion $(t_s, \dot{t}_s)$ satisfying Equation (56), and with the resulting random energy:

$$\mathcal{E}_s = c\, t_s^{-2}\,(2\,\dot{t}_s^2 + 3c - 2) = 2c\,(\dot{t}_s/t_s)^2 + \mathcal{O}(s^{-2}).$$

Let us denote by ζ the explosion time: $\zeta := \sup\{s > 0 \,|\, \dot{t}_s < \infty\} \in]0,\infty]$.

Lemma 5.3.1 *We have almost surely: either* $\lim\limits_{s\to\zeta} \dot{t}_s = 1$ *and* $\zeta = \infty$, *or* $\lim\limits_{s\to\zeta} \dot{t}_s = \infty$.

<u>Proof</u> By Equation (56),

$$\frac{1}{\dot{t}_{s\wedge\zeta}} - c\int_{s_0}^{s\wedge\zeta} \big[1 - \tfrac{1}{\dot{t}_\tau^2}\big]\frac{d\tau}{t_\tau} + \varrho^2 c\int_{s_0}^{s\wedge\zeta} \big[3 + \tfrac{3c+2}{2\,\dot{t}_\tau^2} + \tfrac{3c-2}{\dot{t}_\tau^4}\big]\frac{dt_\tau}{t_\tau^2}$$

is a continuous martingale with quadratic variation $2\varrho^2 c\displaystyle\int_{s_0}^{s\wedge\zeta} \big[1+\tfrac{3c-2}{2\,\dot{t}_\tau^2}\big]\big[1-\tfrac{1}{\dot{t}_\tau^2}\big]\frac{d\tau}{t_\tau^2}$ · Hence, since $\dot{t}_\tau \geq 1$ and therefore $t_\tau \geq \tau$, the process

$$\dot{t}_{s\wedge\zeta}^{-1} - c\int_{s_0}^{s\wedge\zeta} (1 - \dot{t}_\tau^{-2})\,\frac{d\tau}{t_\tau}$$

converges almost surely as $s \to \infty$. As $0 \leq \dot{t}_{s\wedge\zeta}^{-1} \leq 1$, this forces the almost sure convergence of the integral: $\displaystyle\int_{s_0}^{\zeta} \big[1 - \tfrac{1}{\dot{t}_\tau^2}\big]\frac{d\tau}{t_\tau} < \infty$, and of $\dot{t}_{s\wedge\zeta}$, towards some $\dot{t}_\zeta \in [1,\infty]$. Moreover, the convergence of the integral forces either $\zeta < \infty$ and then $\dot{t}_\zeta = \infty$, or $\zeta = \infty$ and then $\dot{t}_\zeta \in \{1,\infty\}$. $\diamond$

The asymptotic behavior can, with positive probability, be partly opposite to that of the preceding R-diffusion:

Proposition 5.3.2 *From any starting point* $(t_{s_0}, \dot{t}_{s_0})$*, there is a positive probability that both* $\mathcal{A}_s = \dot{t}_s$ *and the energy* $\mathcal{E}_s$ *explode. This happens with arbitrary large probability, starting with* $\dot{t}_{s_0}/t_{s_0}$ *sufficiently large and* t_0 *bounded away from zero.*

On the other hand, there is also a positive probability that the hyperbolic angle $\mathcal{A}_s = \dot{t}_s$ *does not explode and goes to 1, and then that the random energy* $\mathcal{E}_s$ *goes to 0. This happens actually with arbitrary large probability, starting with sufficiently large* $t_{s_0}/\dot{t}_{s_0}$.

<u>Proof</u> Let us set $\lambda_s := t_s/\dot{t}_s \geq 0$. From the above proof of Lemma 5.3.1, we get directly:

$$\lambda_{s\wedge\zeta} - \lambda_{s_0} = M_{s\wedge\zeta} + [1+c]\int_{s_0}^{s\wedge\zeta} \big[1 - \tfrac{c}{[1+c]\dot{t}_\tau^2}\big]d\tau - \varrho^2 c\int_{s_0}^{s\wedge\zeta} \big[3 + \tfrac{3c-2}{2\,\dot{t}_\tau^2} + \tfrac{3c-2}{\dot{t}_\tau^4}\big]\frac{d\tau}{\lambda_\tau} \qquad (58)$$

$$\leq M_{s\wedge\zeta} + [1+c](s\wedge\zeta - s_0) - \varrho^2 c \int_{s_0}^{s\wedge\zeta} \left[\frac{3}{\lambda_\tau} - \frac{1+2\,\dot t_\tau^{-2}}{t_\tau\,\dot t_\tau}\right] d\tau\,, \tag{59}$$

(M_s) denoting a martingale having quadratic variation $2\varrho^2 c\int_{s_0}^{s}\big[1+\frac{3c-2}{2\,\dot t_\tau^2}\big]\big[1-\frac{1}{\dot t_\tau^2}\big]d\tau$.

(i) Let us first start the time sub-diffusion from $(t_{s_0},\dot t_{s_0})$ such that $t_{s_0}\geq s_0\geq 1$ and $\dot t_{s_0}\geq n\,m\,t_{s_0}$, with fixed $n\geq 2$ and $m\geq 2+\frac{1+c}{3\varrho^2 c}$, and consider $T:=\zeta\wedge\inf\{s>s_0\,|\,m\,\lambda_s>1\}$. Thus, we have on $[s_0,T]$: $\lambda_\tau^{-1}\geq m$, and then $\frac{3}{\lambda_\tau}-\frac{1+2\,\dot t_\tau^{-2}}{t_\tau\,\dot t_\tau}\geq 3m-3\geq 3+\frac{1+c}{\varrho^2 c}$. Therefore, by Inequality (59), we have almost surely for any $s\geq s_0$:

$$0\leq\ \lambda_{s\wedge T}\leq \lambda_{s_0} - 3\varrho^2 c\,(s\wedge T - s_0) + M_{s\wedge T}\,.$$

Integrating this inequality and letting $s\nearrow\infty$ yields:

$$\tfrac{1}{m}\,\mathbb{P}_{(t_{s_0},\,\dot t_{s_0})}[T<\zeta]\ \leq\ \liminf_{s\to\infty}\ \mathbb{E}[\lambda_{s\wedge T}]\ \leq \lambda_{s_0}\leq \tfrac{1}{nm}\,,\quad \text{whence}\quad \mathbb{P}_{(t_{s_0},\,\dot t_{s_0})}[T=\zeta]\ \geq\ 1-\tfrac{1}{n}\,.$$

Moreover, almost surely on the event $\{T=\zeta\}$, the above inequality

$$0\leq\ \lambda_{s\wedge\zeta}\leq \lambda_{s_0} - 3\varrho^2 c\,(s\wedge\zeta - s_0) + M_{s\wedge\zeta}$$

implies clearly (using that a continuous martingale almost surely cannot go to infinity) $\zeta<\infty$, and by the previous lemma that $\dot t_\zeta=\infty$. Then (58) implies the convergence of $\lambda_{s\wedge\zeta}$ to some $\lambda_\zeta\in\mathbb{R}_+$.
Furthermore, $\dot t_\zeta=\infty$ and $\lambda_\zeta>0$ for finite ζ would imply trivially $t_\zeta=\infty$, whence its logarithmic derivative should explode, which leads to a contradiction.
This proves that we have $\mathbb{P}_{(t_{s_0},\,\dot t_{s_0})}\big[\zeta<\infty\,,\ \lambda_\zeta=0\big]\ \geq\ 1-n^{-1}$.

Since (by the support theorem of Stroock and Varadhan, see for example Theorem 8.1 in [I-W]) from any starting point the sub-diffusion $(t_s,\dot t_s)$ hits with a positive probability some $(t_{s_0},\dot t_{s_0})$ as above, we find there is always a positive probability that $\dot t_s$ and $\mathcal{E}_s$ explode (together).

(ii) Let us now start the time sub-diffusion from $(t_{s_1},\dot t_{s_1})$ such that $n\,m'\,\dot t_{s_1}\leq t_{s_1}$, with fixed $m'\geq 9\varrho^2 c\,(1+c)$ and $n\geq 2+c$, and consider $T':=\zeta\wedge\inf\{s>s_1\,|\,\lambda_s<m'\}$. By Equation (56) we have at once: almost surely, for any $s\geq s_1$,

$$\lambda_{s\wedge T'}^{-1} = \lambda_{s_1}^{-1} + 5\varrho^2 c\int_{s_1}^{s\wedge T'}\big[1+\tfrac{9c-10}{10\,\dot t_\tau^2}\big]\lambda_\tau^{-3}\,d\tau - [1+c]\int_{s_1}^{s\wedge T'}\lambda_\tau^{-2}\,d\tau + c\int_{s_1}^{s\wedge T'} t_\tau^{-2}\,d\tau + M'_{s\wedge T'}\,,$$

(M'_s) denoting a martingale having quadratic variation $2\varrho^2 c\int_{s_1}^{s}\big[1+\frac{3c-2}{2\,\dot t_\tau^2}\big]\big[1-\frac{1}{\dot t_\tau^2}\big]\lambda_\tau^{-4}\,d\tau$.

Since on $[s_1,T']$ we have $\lambda_\tau^{-1}\leq 1/m'$, and then $5\varrho^2 c\big[1+\frac{9c-10}{10\,\dot t_\tau^2}\big]\lambda_\tau^{-1}\leq\frac{5\varrho^2 c\,(1+c)}{m'}\leq 5/9$, we get:

$$0\leq\ \lambda_{s\wedge T'}^{-1}\ \leq\ \lambda_{s_1}^{-1} - c\int_{s_1}^{s\wedge T'}\lambda_\tau^{-2}\,d\tau + \frac{c}{t_{s_1}} + M'_{s\wedge T'}\,.$$

This entails $\int_{s_1}^{T'} \lambda_\tau^{-2}\, d\tau < \infty$ and $\lambda_{s\wedge T'}^{-1} \to \lambda_{T'}^{-1} \in \mathbb{R}_+$ (which implies moreover $\lambda_{T'}^{-1} = 0$ almost surely on $\{T' = \infty\}$), and

$$\tfrac{1}{m'}\, \mathbb{P}_{(t_{s_1},\dot{t}_{s_1})}[T' < \zeta] \le \mathbb{E}_{(t_{s_1},\dot{t}_{s_1})}\big[\lambda_{T'}^{-1}\big] \le \lambda_{s_1}^{-1} + \tfrac{c}{t_{s_1}} \le \tfrac{1+c}{n\,m'}\,.$$

Hence, we get $\mathbb{P}_{(t_{s_1},\dot{t}_{s_1})}[T' = \zeta] \ge 1 - \frac{1+c}{n} > 0$. Furthermore, as in (i) above, $\dot{t}_{T'} = \infty$ and $\lambda_{T'} > 0$ for finite T' is impossible, which excludes $T' = \zeta < \infty$. Therefore

$$\mathbb{P}_{(t_{s_1},\dot{t}_{s_1})}[T' = \zeta = \infty] \ge 1 - \tfrac{1+c}{n} > 0\,.$$

Then from the equation for λ, using that $\big[3 + \frac{9c}{2}\big]\varrho^2 c\, \lambda_\tau^{-1} \le \frac{(6+9c)\varrho^2 c}{2\,m'} < \frac{1}{2}$ on $[s_1, T']$, we get almost surely:

$$\lambda_{s\wedge T'} - \lambda_{s_1} \ge (s\wedge T' - s_1) - \big[3 + \tfrac{9c}{2}\big]\varrho^2 c \int_{s_1}^{s\wedge T'} \frac{d\tau}{\lambda_\tau} + M_{s\wedge T'} \ \ge\ \tfrac{1}{2}\,(s\wedge T' - s_1) + M_{s\wedge T'}\,,$$

which shows (since $[M_s, M_s] = \mathcal{O}(s)$) that almost surely $\{T' = \zeta = \infty\} \subset \{\lambda_s \to \infty\}$. On this same event, by Equation (56) we have almost surely for any $s \ge s_1$:

$$\dot{t}_s - \dot{t}_{s_1'} = \varrho \int_{s_1'}^{s} \sqrt{2c\big[1 + \tfrac{3c-2}{2\,\dot{t}_\tau^2}\big]\big[1 - \tfrac{1}{\dot{t}_\tau^2}\big]}\ \frac{\dot{t}_\tau}{\lambda_\tau}\, dw_\tau + c\int_{s_1'}^{s} \left[5\varrho^2 \frac{1 + \frac{9c-10}{10\,\dot{t}_\tau^2}}{\lambda_\tau^2} - \frac{1 - \dot{t}_\tau^{-2}}{\lambda_\tau}\right] dt_\tau\,,$$

which shows that $\dot{t}_s$ cannot go to infinity, since this would forbid the last integral, and then the right hand side, to go to $+\infty$. Hence, by Lemma 5.3.1, we obtain that almost surely $\{T' = \zeta = \infty\} \subset \{\dot{t}_s \to 1\}$. The proof is ended as in (i) above, by applying the support theorem of Stroock and Varadhan, and by taking n arbitrary large. $\diamond$

6 Sectional relativistic diffusion

We turn now our attention towards a different class of intrinsic relativistic generators on $G(\mathcal{M})$, whose expressions derive directly from the commutation relations of Section 2.2, on canonical vector fields of $TG(\mathcal{M})$. They all project on the unit tangent bundle $T^1\mathcal{M}$ onto a unique relativistic generator $\mathcal{H}^1_{curv}$, whose expression involves the curvature tensor. Semi-ellipticity of $\mathcal{H}^1_{curv}$ requires the assumption of non-negativity of timelike sectional curvatures. Note that in general $\mathcal{H}^1_{curv}$ does not induce the geodesic flow in an empty space.

6.1 Intrinsic relativistic generators on $G(\mathcal{M})$

We shall actually consider among these generators, those which are invariant under the action of $SO(d)$ on $G(\mathcal{M})$. To this aim, we introduce the following dual vertical vector fields, by lifting indexes: $V^{ij} := \eta^{im}\,\eta^{jn}\, V_{mn}$. Note that $V^j \equiv V^{0j} = -V_{0j} = -V_j$, and that $V^{ij} = V_{ij}$ for $1 \le i,j \le d$. We consider again a positive parameter ϱ.

Proposition 6.1.1 *The following four $SO(d)$-invariant differential operators define the same operator $\mathcal{H}^1_{curv}$ on $T^1\mathcal{M}$:*

$$H_0 - \tfrac{\varrho^2}{2}\sum_{j=1}^{d}\Big([H_0,H_j]V^j + V^j[H_0,H_j]\Big)\,;\quad H_0 + \varrho^2\sum_{j=1}^{d}[H_j,H_0]V^j\,;$$

$$H_0 + \varrho^2\sum_{j=1}^{d} R_0^j\, V_j - \varrho^2\sum_{1\le j,k\le d}\mathcal{R}_0{}^{j0k}\, V_jV_k\,;\quad H_0 - \tfrac{\varrho^2}{4}\sum_{1\le i,j\le d}\Big([H_i,H_j]\,V^{ij} + V^{ij}\,[H_i,H_j]\Big)\,;$$

Note that $(\mathcal{H}^1_{curv} - \mathcal{L}^0)$ is self-adjoint with respect to the Liouville measure of $T^1\mathcal{M}$.

The proof will be broken in several lemmas. We begin with the following general and useful computation rules, derived from Sections 2.1 and 2.2.

Lemma 6.1.2 *For $0\le j,k,\ell\le d$, we have :*

$$V^i\mathcal{R}_{ij}{}^{k\ell} = \delta_0^k\, R_j^\ell - \delta_0^\ell\, R_j^k + (1-d)\mathcal{R}_{0j}{}^{k\ell} + \mathcal{R}^\ell{}_{j0}{}^k - \mathcal{R}^k{}_{j0}{}^\ell\,;$$

$$V^i\mathcal{R}_{0i}{}^{k\ell} = \delta_0^\ell\, R_0^k - \delta_0^k\, R_0^\ell\,;\qquad [\,[H_i,H_j],V^i]\, = (d-1)[H_0,H_j]\,.$$

<u>Proof</u> We get the first formula by multiplying by η^{ip} the formula of Lemma 2.2.2, and particularising to $q=0$. As to the second one, by particularising the latter to $j=0$ and changing sign, we get :

$$V^i\mathcal{R}_{0i}{}^{k\ell} = \delta_0^\ell\, R_0^k - \delta_0^k\, R_0^\ell + \mathcal{R}^k{}_{00}{}^\ell - \mathcal{R}^\ell{}_{00}{}^k.$$

Then, we note that $\mathcal{R}^\ell{}_{00}{}^k = \mathcal{R}_0{}^{\ell k}{}_0 = \mathcal{R}^k{}_{00}{}^\ell$. Finally, the last formula derives from the second one and from the commutation relations (4) and (6), as follows:

$$[\,[H_i,H_j],V^i]\, = \tfrac12\,[\mathcal{R}_{ij}{}^{k\ell}\,V_{k\ell},V^i] = \tfrac12\,\mathcal{R}_{ij}{}^{k\ell}\,[V_{k\ell},V^i] - \tfrac12\,(V^i\mathcal{R}_{ij}{}^{k\ell})V_{k\ell}$$

$$= \tfrac12\,\mathcal{R}_{ij}{}^{k\ell}\Big(\delta_k^i V_\ell - \delta_\ell^i V_k + \eta^{ip}\,(\eta_{0\ell}V_{pk} - \eta_{0k}V_{p\ell})\Big) - \Big(\delta_0^k\, R_j^\ell + \tfrac{1-d}{2}\,\mathcal{R}_{0j}{}^{k\ell} + \mathcal{R}^\ell{}_{j0}{}^k\Big)V_{k\ell}$$

$$= R_j^\ell\, V_\ell + \mathcal{R}^p{}_j{}^k{}_0\, V_{pk} - R_j^\ell\, V_\ell + (\tfrac{d-1}{2})\mathcal{R}_{0j}{}^{k\ell}\, V_{k\ell} - \mathcal{R}^\ell{}_{j0}{}^k\, V_{k\ell} = (\tfrac{d-1}{2})\mathcal{R}_{0j}{}^{k\ell}\, V_{k\ell} = (d-1)[H_0,H_j].\ \diamond$$

We get then first the following.

Lemma 6.1.3 *On $C^2(T^1\mathcal{M})$, we have $[\,[H_0,H_j],V^j] = 0$, and*

$$\mathcal{H}^o_{curv} := -\tfrac12\sum_{j=1}^{d}\Big[[H_0,H_j]V^j + V^j[H_0,H_j]\Big] = \sum_{j=1}^{d}[H_j,H_0]V^j = \sum_{j=1}^{d} R_0^j\, V_j - \sum_{1\le j,k\le d} R_0{}^{j0k}\, V_jV_k\,.$$

Proof Using the commutation relations (4) and (6), that $V_{k\ell} = 0$ on $C^2(T^1\mathcal{M})$ for $1 \le k, \ell \le d$, and (10), we have on one hand:

$$[H_0, H_j]V^j \;=\; \tfrac{1}{2}\,\mathcal{R}_{0j}{}^{k\ell}\,V_{k\ell}\,V^j \;=\; \tfrac{1}{2}\,\mathcal{R}_{0j}{}^{k\ell}([V_{k\ell}, V^j] + V^j\,V_{k\ell})$$

$$= \tfrac{1}{2}\,\mathcal{R}_{0j}{}^{k\ell}\,\eta^{ij}\,(\eta_{ik}V_\ell - \eta_{i\ell}V_k + \eta_{0\ell}V_{ik} - \eta_{0k}V_{i\ell}) + \mathcal{R}_{0j}{}^{0\ell}\,V^jV_\ell$$

$$= \;-\mathcal{R}_{0j}{}^{kj}\,V_k + \mathcal{R}_0{}^{ik}{}_0\,V_{ik} + \mathcal{R}_{0j}{}^{0k}\,V^jV_k \;=\; \mathcal{R}_{0j}{}^{0k}\,V^jV_k - R_0^k\,V_k\,.$$

On the other hand, using this first part of proof and Lemma 6.1.2, we get:

$$[[H_0, H_j], V^j] \;=\; \tfrac{1}{2}[\mathcal{R}_{0j}{}^{k\ell}\,V_{k\ell}, V^j] \;=\; \tfrac{1}{2}\,\mathcal{R}_{0j}{}^{k\ell}\,[V_{k\ell}, V^j] - \tfrac{1}{2}(V^j\mathcal{R}_{0j}{}^{k\ell})V_{k\ell}$$

$$= \;-R_0^k\,V_k - \tfrac{1}{2}(\delta_0^\ell\,R_0^k - \delta_0^k\,R_0^\ell)V_{k\ell} \;=\; -R_0^k\,V_k + R_0^k\,V_k \;= 0\,.$$

Using $[H_0, H_j]V^j + V^j[H_0, H_j] = 2\,[H_0, H_j]V^j - [\,[H_0, H_j], V^j]$ ends the proof. $\diamond$

We get then the following.

Lemma 6.1.4 *On $C^2(T^1\mathcal{M})$, we have* $\sum\limits_{1\le i,j\le d}[[H_i, H_j], V^{ij}] = 0\,,$ *and*

$$\sum_{1\le i,j\le d}\Big([H_i, H_j]\,V^{ij} + V^{ij}\,[H_i, H_j]\Big) \;=\; -\,4\,\mathcal{H}^o_{curv}\,.$$

Proof As for the proof of Lemma 6.1.3, we use the commutation relations (4) and (6), that $V_{k\ell} = 0$ on $C^2(T^1\mathcal{M})$ for $1 \le k, \ell \le d$, (10), Lemmas 2.2.2 and 6.1.2, and the symmetries of the Riemann tensor. We have thus on one hand and on $C^2(T^1\mathcal{M})$:

$$[[H_p, H_j], V^{ij}] = \tfrac{1}{2}[\mathcal{R}_{pj}{}^{k\ell}V_{k\ell}, V_{mn}]\,\eta^{im}\eta^{jn} = \tfrac{1}{2}\mathcal{R}_{pj}{}^{k\ell}[V_{k\ell}, V_{mn}]\eta^{im}\eta^{jn} - \tfrac{1}{2}(V_{mn}\,\mathcal{R}_{pj}{}^{k\ell})\eta^{im}\eta^{jn}V_{k\ell}$$

$$= \tfrac{1}{2}(\mathcal{R}_p{}^{ni\ell}V_{\ell n} + R_p^k\,V_k{}^i - \mathcal{R}_p{}^{nki}V_{kn} + \mathcal{R}_p^\ell\,V_\ell{}^i) + \tfrac{1}{2}[(d+1)\mathcal{R}^i{}_p{}^{k\ell} + \eta^{ik}R_p^\ell + \mathcal{R}_p{}^{ki\ell} - \eta^{i\ell}R_p^k - \mathcal{R}_p{}^{\ell ik}]V_{k\ell}$$

$$= \tfrac{1}{2}\Big[-\mathcal{R}_p{}^{ki\ell} + \eta^{i\ell}R_p^k - \mathcal{R}_p{}^{\ell ki} - \eta^{ik}R_p^\ell + (d+1)\mathcal{R}^i{}_p{}^{k\ell} + \eta^{ik}R_p^\ell + \mathcal{R}_p{}^{ki\ell} - \eta^{i\ell}\,R_p^k - \mathcal{R}_p{}^{\ell ik}\Big]V_{k\ell}$$

$$= \;(\tfrac{d+1}{2})\,\mathcal{R}^i{}_p{}^{k\ell}\,V_{k\ell}\,.$$

In particular, we get $[\,[H_i, H_j], V^{ij}] = (\frac{d+1}{2})\,\mathcal{R}^i{}_i{}^{k\ell}\,V_{k\ell} = 0\,.$

And on the other hand, on $C^2(T^1\mathcal{M})$ again:

$$[H_i, H_j]V^{ij} \;=\; \tfrac{1}{2}\,\eta^{im}\eta^{jn}\,\mathcal{R}_{ij}{}^{k\ell}\,V_{k\ell}\,V_{mn} \;=\; \eta^{i0}\eta^{jn}\,\mathcal{R}_{ij}{}^{k\ell}\,V_{k\ell}\,V_n \;=\; \mathcal{R}_0{}^{jk\ell}\,V_{k\ell}\,V_j$$

$$= \;\mathcal{R}_0{}^{jk\ell}([V_{k\ell}, V_j] + V_j\,V_{k\ell}) \;=\; \mathcal{R}_0{}^{jk\ell}(\eta_{jk}V_\ell - \eta_{j\ell}V_k + \eta_{0\ell}V_{jk} - \eta_{0k}V_{j\ell}) + \mathcal{R}_0{}^{jk\ell}\,V_j\,V_{k\ell}$$

$$= -2\,R_0^k\,V_k + 2\,\mathcal{R}_0{}^{jk}{}_0\,V_{jk} + 2\,\mathcal{R}_0{}^{j0\ell}\,V_jV_\ell \;=\; 2\,(\mathcal{R}_0{}^{j0k}\,V_jV_k - R_0^k\,V_k) \;=\; -2\,\mathcal{H}^o_{curv}\,. \;\diamond$$

The final assertion relating to the Liouville measure is proved as in Theorem 3.2.1.

Proposition 6.1.5 *In local coordinates, the second order operator $\mathcal{H}^1_{curv}$ defined on $T^1\mathcal{M}$ by Proposition 6.1.1 reads :*

$$\mathcal{H}^1_{curv} \;=\; \dot\xi^j \frac{\partial}{\partial \xi^j} - \dot\xi^i\dot\xi^j\, \Gamma^k_{ij}\, \frac{\partial}{\partial \dot\xi^k} + \tfrac{\varrho^2}{2}\, \dot\xi^n\, \tilde R^k_n\, \frac{\partial}{\partial \dot\xi^k} - \tfrac{\varrho^2}{2}\, \dot\xi^p \dot\xi^q\, \widetilde{\mathcal{R}}_p{}^k{}_q{}^\ell\, \frac{\partial^2}{\partial \dot\xi^k \partial \dot\xi^\ell}$$

$$= \dot\xi^j \frac{\partial}{\partial \xi^j} - \dot\xi^i\dot\xi^j\, \Gamma^k_{ij}\, \frac{\partial}{\partial \dot\xi^k} + \tfrac{\varrho^2}{2}\, \dot\xi^m\, \widetilde{\mathcal{R}}_{mnpq} \left(g^{nq}\, g^{pk}\, \frac{\partial}{\partial \dot\xi^k} - \dot\xi^p\, g^{nk} g^{q\ell}\, \frac{\partial^2}{\partial \dot\xi^k \partial \dot\xi^\ell}\right).$$

<u>Proof</u> By Section 2.3, we have on $C^2(T^1\mathcal{M})$:

$$V_j V_k \;=\; e^n_j e^\ell_k\, \frac{\partial^2}{\partial e^n_0 \partial e^\ell_0} + \delta_{jk}\, e^n_0\, \frac{\partial}{\partial e^n_0} + e^n_0 e^\ell_0\, \frac{\partial^2}{\partial e^n_j \partial e^\ell_k} + e^n_j\, \frac{\partial}{\partial e^n_k} \;=\; e^n_j e^\ell_k\, \frac{\partial^2}{\partial e^n_0 \partial e^\ell_0} + \delta_{jk}\, e^n_0\, \frac{\partial}{\partial e^n_0}\,,$$

whence by Lemma 6.1.3 :

$$\mathcal{H}^o_{curv} \;=\; -\mathcal{R}_0{}^{j0k}\, e^n_j e^\ell_k\, \frac{\partial^2}{\partial e^n_0 \partial e^\ell_0} - \mathcal{R}_0{}^{j0k}\, \delta_{jk}\, e^n_0\, \frac{\partial}{\partial e^n_0} + \sum_{j=1}^{d} R^j_0\, e^n_j\, \frac{\partial}{\partial e^n_0}$$

$$= -\mathcal{R}_0{}^{j0k}\, e^n_j e^q_k\, \frac{\partial^2}{\partial e^n_0 \partial e^q_0} + R^j_0\, e^n_j\, \frac{\partial}{\partial e^n_0} \quad \text{(including now } j = 0)\,.$$

On the other hand, by Formula (18) we have :

$$\mathcal{R}_0{}^{j0k} = \mathcal{R}_{0abc}\, \eta^{aj} \eta^{b0} \eta^{ck} = e^m_0\, e^n_a\, \widetilde{\mathcal{R}}_{mnp\ell}\, e^p_b\, e^\ell_c\, \eta^{aj}\, \eta^{b0}\, \eta^{ck} = e^m_0\, e^p_0\, e^r_a\, \widetilde{\mathcal{R}}_{mrp\ell}\, e^\ell_c\, \eta^{aj}\, \eta^{ck},$$

whence

$$\mathcal{R}_0{}^{j0k}\, e^n_j\, e^q_k = e^m_0\, e^p_0\, e^r_a\, \widetilde{\mathcal{R}}_{mrp\ell}\, e^\ell_c\, \eta^{aj}\, \eta^{ck}\, e^n_j\, e^q_k = e^m_0\, e^p_0\, \widetilde{\mathcal{R}}_{mrp\ell}\, g^{rn}\, g^{q\ell} = e^m_0\, e^p_0\, \widetilde{\mathcal{R}}_m{}^n{}_p{}^q.$$

And in a similar way, by (16) :

$$R^j_0\, e^n_j = e^m_0\, e^q_i\, \tilde R_{mq}\, \eta^{ij}\, e^n_j = e^m_0\, \tilde R^n_m = e^m_0\, \widetilde{\mathcal{R}}_{m\ell pq}\, g^{\ell q}\, g^{pn}.$$

This and (13),(14) yield the wanted formula, whose coefficients depend only on $(\xi, \dot\xi) \in T^1\mathcal{M}$, as it must be by the $SO(d)$-invariance underlined in Proposition 6.1.1. $\diamond$

6.2 Sign condition on timelike sectional curvatures

The generator $\mathcal{H}^1_{curv}$ defined on $T^1\mathcal{M}$ by Proposition 6.1.1 is covariant with any Lorentz isometry of $(\mathcal{M}, g)$. Hence, it is a candidate to generate a covariant "sectional" relativistic diffusion on $T^1\mathcal{M}$, provided it be semi-elliptic.

As a consequence of Section 6.1, the intrinsic sectional generator we are led to consider on $T^1\mathcal{M}$ is $\mathcal{H}^1_{curv}$, restriction of $H_0 + \frac{\varrho^2}{2}\big(R^j_0\, V_j - R_0{}^{j0k}\, V_j V_k\big)$.

Now, a necessary and sufficient condition, in order that such an operator be the generator of a well-defined diffusion, is that it be subelliptic.

We are thus led to consider the following negativity condition on the curvature:

$$\big\langle \mathcal{R}(u\wedge v), u\wedge v\big\rangle_\eta \leq 0\,, \quad \text{for any timelike } u \text{ and any spacelike } v\,. \tag{60}$$

This condition is equivalent to the lower bound on sectional curvatures of timelike planes $\mathbb{R}u+\mathbb{R}v$:

$$\frac{\langle \mathcal{R}(u\wedge v), u\wedge v\rangle_\eta}{g(u\wedge v, u\wedge v)} \,\geq\, 0\,,$$

since $g(u\wedge v, u\wedge v) := g(u,u)g(v,v)-g(u,v)^2 < 0$ for such planes.

Note that Sectional Curvature has proved to be a natural tool in Lorentzian geometry, see for example [H], [H-R].

We test this negativity condition on warped products, in Corollary 6.2.1 below. When this negativity condition is fulfilled, we call the resulting covariant diffusion on $T^1\mathcal{M}$, which has generator $\mathcal{H}^1_{curv}$ given by Propositions 6.1.1, 6.1.5, the <u>sectional relativistic diffusion</u>.

Corollary 6.2.1 *Consider a Lorentz manifold $(\mathcal{M},g)$ having the warped product form. Then the sign condition (60) is equivalent to: $\alpha''\leq 0$ on I, together with the following lower bound on sectional curvatures of the Riemannian factor (M,h):*

$$\inf_{X,Y\in TM} \frac{\langle \mathcal{K}\,(X\wedge Y)\,, X\wedge Y\rangle}{h(X,X)\,h(Y,Y)-h(X,Y)^2} \,\geq\, \sup_I\{\alpha\,\alpha''-\alpha'^{\,2}\}. \tag{61}$$

<u>Proof</u> Let us denote by $SR(U\wedge V)$ the sectional curvature of the timelike plane associated with $U\wedge V$. Recall from Section 6.2 that the negativity condition (60) reads simply $SR(U\wedge V)\geq 0$, for any timelike U and spacelike V. By choosing a pseudo-orthonormal basis of such given timelike plane, we can moreover restrict to $g(U,U)=1=-g(V,V)$ and $g(U,V)=0$. Setting $U=u\partial_t+X$ and $V=v\partial_t+Y$, with $u,v\in C^0(I)$ and $X,Y\in TM$, we can thus suppose that:

$$u=\sqrt{\alpha^2\,h(X,X)+1}\,;\quad v=\sqrt{\alpha^2\,h(Y,Y)-1}\,;\quad uv=\alpha^2\,h(X,Y)\,,$$

which implies $\alpha^2\,h(X\wedge Y, X\wedge Y)+h(Y,Y)-h(X,X)=\alpha^{-2}$, and

$$h(uX-vY, uX-vY)=\alpha^2\,h(X\wedge Y, X\wedge Y)+\alpha^{-2}\,.$$

Recall that $h(X\wedge Y, X\wedge Y) := h(X,X)h(Y,Y)-h(X,Y)^2$. Now, by (36), this entails:

$$SR(U\wedge V)=\alpha^2\,\langle \mathcal{K}\,(X\wedge Y)\,, X\wedge Y\rangle - \alpha\alpha''[\alpha^2\,h(X\wedge Y, X\wedge Y)+\alpha^{-2}]+|\alpha\alpha'|^2\,h(X\wedge Y, X\wedge Y)$$

$$=\alpha^2\Big[\langle \mathcal{K}\,(X\wedge Y)\,, X\wedge Y\rangle-(\alpha\,\alpha''-\alpha'^{\,2})\,h(X\wedge Y, X\wedge Y)\Big]-\alpha''/\alpha\,.$$

Reciprocally, given any $X, Y \in TM$ such that $\alpha^2\, h(X \wedge Y, X \wedge Y) + h(Y,Y) - h(X,X) = \alpha^{-2}$ and $h(Y,Y) \geq \alpha^{-2}$, setting $u := \sqrt{\alpha^2\, h(X,X) + 1}$ and $v := \pm\sqrt{\alpha^2\, h(Y,Y) - 1}$, we get $(uv)^2 = \alpha^4\, h(X,Y)^2$, and thence a basis (U,V) as above.

Hence, the negativity condition (60) is equivalent to:

$$0 \leq \alpha^2 \Big(\langle \mathcal{K}\,(X \wedge Y)\,, X \wedge Y \rangle - (\alpha\, \alpha'' - \alpha'^{\,2})\, h(X \wedge Y, X \wedge Y) \Big) - \alpha''/\alpha\,,$$

for any $X, Y \in TM$ such that $\alpha^2\, h(X \wedge Y, X \wedge Y) + h(Y,Y) - h(X,X) = \alpha^{-2}$ and $h(Y,Y) \geq \alpha^{-2}$.

Distinguishing between collinear and non-collinear pairs X, Y, and denoting in the latter case by $SK(X \wedge Y)$ the sectional curvature of the plane associated with $X \wedge Y$, this condition splits into both: $(\alpha''/\alpha) \leq 0$, together with:

$$\frac{(\alpha''/\alpha)}{\alpha^2\, h(X \wedge Y, X \wedge Y)} \;\leq\; SK(X \wedge Y) - (\alpha\, \alpha'' - \alpha'^{\,2})\,,$$

for any non-collinear $X, Y \in TM$ such that $\alpha^2\, h(X \wedge Y, X \wedge Y) + h(Y,Y) - h(X,X) = \alpha^{-2}$ and $h(Y,Y) \geq \alpha^{-2}$. We shall have proved that this condition is equivalent to the wanted inequality (61), if we show now that for any given $\alpha > 0$, any given plane P in TM is generated by pairs X, Y such that $\alpha^2\, h(X \wedge Y, X \wedge Y) + h(Y,Y) - h(X,X) = \alpha^{-2}$, $h(Y,Y) \geq \alpha^{-2}$, and such that $h(X \wedge Y, X \wedge Y)$ is arbitrary large.

Now, starting from an arbitrary $\{X_0, Y_0\}$ generating P, take $Y := \frac{Y_0}{\alpha\sqrt{h(Y_0,Y_0)}}$ and $X := q\Big[X_0 - \frac{h(X_0,Y_0)}{h(Y_0,Y_0)}\, Y_0\Big]$. Then $\alpha^2\, h(X \wedge Y, X \wedge Y) + h(Y,Y) - h(X,X) = \alpha^{-2} = h(Y,Y)$, and $h(X \wedge Y, X \wedge Y) = q^2\, \frac{h(X_0 \wedge Y_0, X_0 \wedge Y_0)}{\alpha^2\, h(Y_0,Y_0)}$ is indeed arbitrary large, for arbitrary large q.
$\diamond$

In particular, in an Einstein-de Sitter-like manifold, the sign condition (60) holds if and only if $\alpha'' \leq 0$, i.e. if and only if $c \leq 1$.

6.3 Sectional diffusion in an Einstein-de Sitter-like manifold

We must have here $c \leq 1$. By (40), we have: for $0 \leq k \leq d$ and $1 \leq m, n, p, q \leq d$,

$$\widetilde{\mathcal{R}}_{0nkq} = \delta_{0k}\, c\,(c-1)\, t^{2c-2}\, h_{nq} \qquad \text{and} \qquad \widetilde{\mathcal{R}}_{mnpq} = c^2\, t^{4c-2}\, [h_{mq}\, h_{np} - h_{mp}\, h_{nq}] - t^{2c}\, \tilde{K}_{mnpq}\,.$$

Using Cartesian coordinates (x^j) instead of spherical coordinates (r, φ, ψ) for the Euclidian factor $M = \mathbb{R}^3$, we have merely:

$$\widetilde{\mathcal{R}}_{0nkq} = c\,(c-1)\, t^{2c-2}\, \delta_{0k}\, \delta_{nq} \qquad \text{and} \qquad \widetilde{\mathcal{R}}_{mnpq} = c^2\, t^{4c-2}\, [\delta_{mq}\, \delta_{np} - \delta_{mp}\, \delta_{nq}]\,.$$

Thence, for $0 \leq k \leq d$ and $1 \leq m, n, p, q \leq d$:

$\widetilde{\mathcal{R}}_0{}^n{}_k{}^q = c(c-1)\, t^{-2c-2}\, \delta_{0k}\, \delta^{nq}$ and $\widetilde{\mathcal{R}}_m{}^n{}_p{}^q = c(c-1)\, t^{2c-2}\, \delta_{mp}\, \delta^{tn}\, \delta^{tq} + c^2\, t^{-2}\, [\delta^q_m\, \delta^n_p - \delta_{mp}\, \delta^{nq}]$.

And $\quad \tilde{R}_{ij} = \delta_{ij}\left(3c\,(1-c)\,t^{-2}\delta_{0j} + c\,(3c-1)\,t^{2c-2}[1-\delta_{0j}]\right), \quad$ whence

$$\tilde{R}^k_n = -\delta^k_n\, c\, t^{-2}\left(3(c-1)\delta_{0n} + (3c-1)[1-\delta_{0n}]\right).$$

And by (39), the non-vanishing Christoffel coefficients are:

$$\Gamma^k_{0j} = \Gamma^k_{j0} = c\,t^{-1}\,\delta^k_j\,, \quad \text{and} \quad \Gamma^0_{ij} = c\,t^{2c-1}\,\delta_{ij}\,, \quad \text{for } 1\le i,j,k\le d\,.$$

Therefore, by Proposition 6.1.5, we have:

$$\mathcal{H}^1_{curv} = \dot{\xi}^j\frac{\partial}{\partial \xi^j} - \dot{\xi}^i\dot{\xi}^j\,\Gamma^k_{ij}\,\frac{\partial}{\partial\dot{\xi}^k} + \tfrac{\varrho^2}{2}\,\dot{\xi}^n\tilde{R}^k_n\,\frac{\partial}{\partial\dot{\xi}^k} - \tfrac{\varrho^2}{2}\,\dot{\xi}^p\dot{\xi}^q\,\widetilde{\mathcal{R}}_p{}^k{}_q{}^\ell\,\frac{\partial^2}{\partial\dot{\xi}^k\partial\dot{\xi}^\ell}$$

$$= \dot{\xi}^j\frac{\partial}{\partial\xi^j} - \frac{c}{t}(\dot{t}^2-1)\frac{\partial}{\partial\dot{t}} - \frac{2c}{t}\,\dot{t}\,\dot{x}^j\frac{\partial}{\partial\dot{x}^j} - \frac{3\varrho^2 c}{2\,t^2}(c-1)\,\dot{t}\,\frac{\partial}{\partial\dot{t}} - \frac{\varrho^2 c}{2\,t^2}(3c-1)\,\dot{x}^j\frac{\partial}{\partial\dot{x}^j}$$

$$-\frac{\varrho^2 c\,(c-1)}{2\,t^2}(\dot{t}^2-1)\frac{\partial^2}{\partial\dot{t}^2} - \frac{\varrho^2 c\,(c-1)}{2\,t^{2c+2}}\,\dot{t}^2\,\Delta_x + \frac{\varrho^2c^2}{2\,t^2}\left[t^{-2c}(\dot{t}^2-1)\Delta_x - \dot{x}^i\dot{x}^j\,\frac{\partial^2}{\partial\dot{x}^i\partial\dot{x}^j}\right]$$

$$= \dot{\xi}^j\frac{\partial}{\partial\xi^j} - \frac{c}{t}(\dot{t}^2-1)\frac{\partial}{\partial\dot{t}} - \frac{3\varrho^2 c}{2\,t^2}(c-1)\,\dot{t}\,\frac{\partial}{\partial\dot{t}} - \frac{2c}{t}\,\dot{t}\,\dot{x}^j\frac{\partial}{\partial\dot{x}^j} - \frac{\varrho^2 c\,(3c-1)}{2\,t^2}\,\dot{x}^j\frac{\partial}{\partial\dot{x}^j}$$

$$+\frac{\varrho^2 c\,(1-c)}{2\,t^2}(\dot{t}^2-1)\frac{\partial^2}{\partial\dot{t}^2} + \frac{\varrho^2 c}{2\,t^{2c+2}}\,(\dot{t}^2-c)\,\Delta_x - \frac{\varrho^2c^2}{2\,t^2}\,\dot{x}^i\dot{x}^j\,\frac{\partial^2}{\partial\dot{x}^i\partial\dot{x}^j}\,.$$

We see that even in this simple case, the sectional and curvature diffusion differ tangibly, apart from the fact that the range of values of c for which they are defined are different.

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